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MOMENTUM AND ENERGY IN GENERAL RELATIVITY AND GRAVITATIONAL RADIATION

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CONTENTS

Synopsis

The energy-momentum complex, which was formulated in terms of tetrad variables in an earlier paper in Mat. Fys. Skr., is applied to the exact asymptotic solution of Einstein's field equations for an axi-symmetric system given by Bondi and his collaborators. The formulae derived for the gravitational energy radiated per unit time and for the total energy of the system at any time confirm a conjecture by Bondi. The transformation properties of the total momentum and energy for a non-closed system under asymptotic Lorentz transformations are derived and the approximate plane gravitational waves at large distances from a radiating system are investigated. As regards energy and momentum, such waves are closely analogous to electromagnetic waves emitted by a system of accelerated electrically charged particles.

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1. Introduction and Summary

Since the first years of Einstein's theory of gravitation the question whether or not a system of accelerated massive bodies loses energy by emission of gravitational radiation has given rise to many controversial discussions. The main reasons for this somewhat unusual situation in physics are the following. On account of the non-linear character of Einstein's field equations it is difficult to find sufficiently general exact solutions of these equations and most of the discussions on gravitational radiation have therefore been based on solutions of the "linearized" field equations. However, in many cases, such solutions have been shown to be good approximations to the solutions of the exact field equations only over limited parts of space and it has been doubted whether the results obtained by means of these solutions can be fully trusted. Moreover, until recently one did not have a consistent expression for the gravitational energy current which, in analogy with Poynting's theorem, could be used for calculating the amount of energy carried away by the gravitational waves.

It is well known that the energy-momentum complex $\Theta_i^{\,k}$ given by Einstein many years ago does not allow to calculate the distribution of the energy and the energy flux in a physically satisfactory way, since the result depends on the spatial coordinates used. But even if one is interested only in the total energy and its possible variation in time, such as in calculations of the energy emission from an insular system, the complex $\Theta_i^{\,k}$ is applicable only in special systems of coordinates. In the trivial case of a completely empty space, for instance, Einstein's expression gives an infinite value for the total energy when calculated in polar coordinates, in contrast to the correct value zero obtained if one uses Cartesian coordinates. This means, strictly speaking, that this expression is *not* in accordance with the general principle of relativity according to which all relations between measurable physical quantities, such as the total energy and the components of the metric tensor, must have the same form in all systems of space-time coordinates.

The difficulties mentioned above have now been overcome. In a recent, most interesting paper, BONDI et al.[1] have been able to give the exact form of the metric at large spatial distances from an axi-symmetric, but otherwise arbitrary, insular system of matter that emits gravitational waves into the surrounding empty space, and, in a paper from 1961[2], we arrived at an expression T_i^k for the energy-momentum complex which is in accordance with the principle of relativity and which therefore meets the objections raised against Einstein's expression Θ_i^k. In the present paper, the complex T_i^k is applied to the solutions of BONDI et al. Thereby we obtain consistent expressions for the total momentum and energy as well as for the time variations of these quantities in the case of an arbitrary axi-symmetric system emitting gravitational waves. Some of the results of these calculations have been published previously in a note in Physics Letters[3].

In section 2, we give an outline of the basic theory and a survey of earlier results as well as some new results regarding the energy-momentum complex. In contrast to the complex Θ_i^k which can be expressed directly in terms of the metric components and its derivatives, the complex T_i^k is given directly in terms of tetrad fields which are determined by the metric only up to arbitrary Lorentz rotations of the tetrads. T_i^k is not invariant under such rotations. However, as will be shown in detail in section 6, the values of the total energy and momentum obtained by means of T_i^k are invariant under all Lorentz rotations of the tetrads which are in accordance with the boundary conditions formulated in section 2.

Section 3 contains a survey of the main results obtained by BONDI et al. in[1] and it is shown that Einstein's expression Θ_i^k gives unreasonable results for the energy radiation and the total energy in the system of coordinates adopted in[1]. In contrast to this result it is shown, in section 4, that the complex T_i^k gives a consistent value for the energy radiation, which confirms a conjecture by BONDI regarding the total energy radiated by an axi-symmetric system. In addition to that, the intensity of the energy radiation in different directions is determined.

The total energy and momentum at any time are defined and calculated in section 5 and, as regards the energy, the result confirms a conjecture by BONDI in part D of [1]. The change in the total momentum per unit time is shown to correspond to a recoil effect of the emitted gravitational radiation of the same kind as for emission of photons.

In section 7, we investigate the transformation properties of the total momentum and energy of the matter system as well as of the emitted radiation under asymptotically Lorentzian transformations. Finally, it is shown in section 8 that the gravitational radiation at large distances from the system has the form of approximate plane waves with an everywhere positive energy density and a momentum density equal to the energy current density divided by c^2 like in the case of a plane electromagnetic wave. Details of the calculations are collected in the Appendix.

2. The Energy-Momentum Complex

In general relativity the energy and momentum of the complete system of matter plus gravitational field is described by an energy-momentum complex of the form

$$\mathsf{T}_i^{\;k} = \mathfrak{T}_i^{\;k} + \mathsf{t}_i^{\;k}. \tag{2.1}$$

Here, $\mathfrak{T}_i^{\;k}$ is the energy-momentum tensor density of the matter, which is a function of the matter field variables *and* the gravitational variables, while the complex $\mathsf{t}_i^{\;k}$ of the gravitational field is an algebraic function of the gravitational field variables only. $\mathfrak{T}_i^{\;k}$ also appears as the source of the gravitational field in Einstein's field equations

$$\mathfrak{G}_i^{\;k} \equiv \mathfrak{R}_i^{\;k} - \frac{1}{2}\,\delta_i^{\;k}\,\mathfrak{R} = -\varkappa\,\mathfrak{T}_i^{\;k} \tag{2.2}$$

which determine the metric for a given matter distribution. If we eliminate $\mathfrak{T}_i^{\;k}$ in (2.1) by means of the field equations, the complex $\mathsf{T}_i^{\;k}$ appears as a function of the gravitational field variables only.

A satisfactory solution of the energy problem in general relativity requires that the energy-momentum complex satisfies the following conditions:

1. $\mathsf{T}_i^{\;k}(x)$ is an afine tensor density depending algebraically on the gravitational field variables and their derivatives of the first and second orders and it satisfies the divergence relation

$$\mathsf{T}_{i\;\;,\,k}^{\;k} \equiv \frac{\partial \mathsf{T}_i^{\;k}}{\partial x^k} = 0\,. \tag{2.3}$$

2. A matter system for which the metric asymptotically at large spatial distances from the system is of the Schwarzschild type is called a *closed*

system. In this case we can use coordinates which are asymptotically rectilinear and then we must require that the quantities*

$$P_i = \iiint\limits_{x^4 = \text{const.}} \mathsf{T}_i^{\ 4}\, dx^1\, dx^2\, dx^3 \tag{2.4}$$

are constant in time and that they transform as the covariant components of a free vector under linear space-time transformations. This property is essential for the interpretation of $P_i = \{P_\iota - H\}$ as the total momentum and energy vector.

3. $\mathsf{T}^k \equiv \mathsf{T}_4^{\ k}$ is transformed like a 4-vector density under the group of purely spatial transformations

$$\overline{x}^\iota = f^\iota(x^\varkappa), \quad \overline{x}^4 = x^4 \tag{2.5}$$

which leave the time scale and the system of reference unchanged. This property makes the "energy content of any volume of space V", i.e.

$$H_V = -\iiint\limits_V \mathsf{T}_4^{\ 4}\, dx^1\, dx^2\, dx^3 = -\iiint\limits_V \overline{\mathsf{T}}_4^{\ 4}\, d\overline{x}^1\, d\overline{x}^2\, d\overline{x}^3 \tag{2.6}$$

independent of the spatial coordinates used in the evaluation of the integral. Thus, *3.* is the condition of localizability of the energy in a gravitational field.

The classical expression for the energy-momentum complex given by Einstein many years ago [(4)] is of the form

$$\Theta_i^{\ k} = \mathfrak{T}_i^{\ k} + \vartheta_i^{\ k}. \tag{2.7}$$

Here, $\vartheta_i^{\ k}$ is a homogeneous quadratic function of the first-order derivatives of the metric tensor which is obtained from the Lagrangian

$$\mathfrak{L}_E = \sqrt{-g}\, g^{ik}\left(\Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^l_{im}\Gamma^m_{kl}\right) \tag{2.8}$$

by the equation

$$\vartheta_i^{\ k} = \frac{1}{2\varkappa}\left\{\frac{\partial \mathfrak{L}_E}{\partial g^{lm}{}_{,k}}\, g^{lm}{}_{,i} - \delta_i^k\, \mathfrak{L}_E\right\}. \tag{2.9}$$

* In the following, we shall use natural units in which the velocity of light c and Newton's gravitational constant k are equal to one. Consequently, Einstein's constant $\varkappa$ has the value 8π. Further, Latin indices run from 1 to 4, Greek indices from 1 to 3, and the signature is $\varepsilon_i = \{1, 1, 1, -1\}$.

The explicit expression for $\vartheta_i^{\,k}$ is

$$\vartheta_i^{\,k} = \frac{1}{2\varkappa}\{\Gamma^k_{lm}(\sqrt{-g}\,g^{lm})_{,i} - \Gamma^l_{ml}(\sqrt{-g}^{km})_{,i} - \delta_i^{\,k}\,\mathfrak{L}_E\}. \tag{2.10}$$

If we eliminate $\mathfrak{T}_i^{\,k}$ in (2.7) by means of the field equations (2.2), the complex $\Theta_i^{\,k}$ can be written, as was first shown by v. FREUD(5), in the form

$$\Theta_i^{\,k} = h_i^{\,kl}{}_{,\,l} \tag{2.11}$$

with the superpotential

$$h_i^{\,kl} = -h_i^{\,lk} = \frac{g_{in}}{2\varkappa\sqrt{-g}}[(-g)(g^{kn}g^{lm} - g^{ln}g^{km})]_{,m}. \tag{2.12}$$

As is well known, Einstein's expression $\Theta_i^{\,k}$ satisfies the conditions *1.* and *2.*, but *not* the localizability condition *3.* Therefore, EINSTEIN came to the conclusion that, in general relativity, the energy content H_V of a finite part of space has no exact physical meaning.

Only the total energy

$$H_E = -\iiint \Theta_4^{\,4}\,dx^1\,dx^2\,dx^3, \tag{2.13}$$

obtained by integrating over the whole 3-dimensional space with $t = x^4 =$ const., should have a well-defined physical meaning. It has been argued that this is quite natural, since it is difficult to imagine how one could measure the energy contained in a small part of the system. On the other hand, the total energy is certainly a measurable quantity, since the total mass can be measured, for instance, by weighing the system on a balance or by measuring its reaction under the influence of external forces. Therefore, it would seem that Einstein's point of view is in accordance with the nature of the problem.

Nevertheless, one may have some doubts as to the validity of the expression (2.13) for the total energy, since it is *not* in accordance with the general principle of relativity. According to this principle, *any relation between measurable physical quantities,* such as the total energy or mass and the components of the metric tensor, *must have the same form in any system of coordinates.* In other words, we can only trust an expression like (2.13) if it represents the energy in any system of coordinates, and this is obviously not the case. Take, for instance, two systems of coordinates connected by a purely spatial transformation (2.5); then, the total energy must certainly have the same value in these two systems, for the result of an experiment, which allows to determine the total mass, is of course completely independent of the way we choose to name the different points in space. However, since the complex $\Theta_i^{\ k}$ does not satisfy the condition *3.*, the equation (2.13) gives in general quite different values for the total energy in the two systems of coordinates of the type considered.

As was pointed out long ago by BAUER(6), this holds even for the trivial case of a completely empty space where space-time is flat. In Cartesian coordinates, H_E is here zero as it should be, but if we use the metric corresponding to polar coordinates in the evaluation of H_E, we get an entirely different result. In fact, the integral in (2.13) is divergent in this case, and it should be noted that the divergence arises from the large distances r and not from the singular point $r = 0$. A similar situation we meet in the case of an arbitrary physical system, and this cannot be considered satisfactory in a general theory of relativity.

The importance of the restricted group of transformations (2.5) lies only in the fact that we can be sure that the total energy must be unchanged under these transformations. For a more general transformation where the time scale and the motion of the frame of reference are changed, as for instance for a simple Lorentz transformation, we must in general expect a change in the total energy of the physical system. The invariance of the total energy under the transformations (2.5) requires that the energy-momentum complex satisfies also the condition *3.* At first sight, one might think that the condition *3.* is too stringent if we give up the idea of localizability and only regard the *total* energy as a measurable quantity, for with *3.* the equation (2.6) is valid for any finite volume V and not only for V equal to the whole 3-dimensional space. However, it should be remembered that the system of coordinates in a given system of reference often consists of an "atlas" of different overlapping local maps, inside which the components of the metric tensor are regular (7), and it is then essential that the equation (2.6) holds for any volume V which lies inside a region of overlapping of two coordinate patches.

It seems therefore that a satisfactory solution of the energy problem in accordance with the general principle of relativity requires the existence of an energy-momentum complex with all the properties *1.–3.* Now, it can be shown (8) that, if the gravitational variables are taken to be the components of the metric tensor, the only complex which satisfies the conditions *1.* and *2.* is Einstein's expression $\Theta_i^{\ k}$, and it is thus impossible also to have *3.* satisfied. Therefore, it seems that we are in a hopeless situation. However, gravitational fields may also be described by so-called tetrad fields instead of by the metric tensor. There are even certain matter systems where one has to use a tetrad description of the gravitational field. This holds, for instance, in the case of a fermion field under the influence of a gravitational field, where the latter has to be described by a tetrad field. In fact, in the usual generally covariant form of the Dirac equation [9], the gravitational field is represented by a tetrad field and not directly by the metric. It is therefore natural to assume that the tetrad field variables are the fundamental gravitational variables and, as was shown in reference 2, with this assumption it is possible to define an energy-momentum complex which satisfies all the conditions *1.–3.*

Let $h^{(a)}{}_i$ be the covariant components of the a'th tetrad vector which is space-like for $a = 1, 2, 3$ and time-like for $a = 4$. Further, let us put

$$h_{(a)\,i} = \eta_{(ab)}\, h^{(b)}{}_i, \tag{2.14}$$

where $\eta_{(ab)}$ is the constant diagonal matrix with the diagonal elements $\{1, 1, 1, -1\}$. Then, the connection between the tetrad field and the metric field at every point is given by

$$h^{(a)}{}_i\, h_{(a)\,k} = g_{ik} \tag{2.15}$$

$$h^{(a)}{}_i\, h_{(b)}{}^i = \delta^a_b. \tag{2.16}$$

Further, we have

$$\sqrt{-g} = |\,h\,|, \tag{2.17}$$

where $h = \det\{h_{(a)\,i}\}$ is the determinant with $h_{(a)\,i}$ in the a'th row and i'th column.

The starting point of the developments in [2] was the remark that the curvature scalar density $\mathfrak{R}$, when expressed in terms of the tetrad field by means of (2.15), takes the form

$$\mathfrak{R} = \mathfrak{L} + \mathfrak{h}, \tag{2.18}$$

where $\mathfrak{h}$ has the form of a usual divergence, which is of no importance in the variational principle, and

$$\mathfrak{L} = |\,h\,|\,[h_{(a)}{}^r{}_{;s}\, h^{(a)\,s}{}_{;r} - h_{(a)}{}^r{}_{;r}\, h^{(a)\,s}{}_{;s}]. \tag{2.19}$$

Here, the semicolon means covariant differentiation so that the Langrangian $\mathfrak{L}$ is a scalar density under arbitrary space-time transformations (a true scalar density) in contrast to the Lagrangian $\mathfrak{L}_E$ in (2.8) which has this property under linear transformations only. Further, since the Christoffel symbols by (2.15) are seen to be linear functions of the first-order derivatives of the tetrad variables, the same holds for $h_{(a)\ ;\, s}^{\ \ r}$. Hence, $\mathfrak{L}$ (just as $\mathfrak{L}_E$) has the important property of being a homogeneous quadratic function of the first-order derivatives of the gravitational field variables.

The energy-momentum complex $\mathsf{T}_i^{\ k}$ which, in [2], was shown to satisfy the conditions *1.–3.*, is

$$\mathsf{T}_i^{\ k} = \mathfrak{T}_i^{\ k} + \mathfrak{t}_i^{\ k}$$

with

$$\left.\begin{aligned} \mathfrak{t}_i^{\ k} \equiv \sqrt{-g}\, t_i^{\ k} &= \frac{1}{2\varkappa}\left[\frac{\partial \mathfrak{L}}{\partial h^{(a)\,l}_{\ \ \ ,\,k}}\, h^{(a)\,l}_{\ \ \ ,\,i} - \delta_i^{\ k}\, \mathfrak{L}\right] \\ &= \frac{1}{2\varkappa}\left[\frac{\partial \mathfrak{L}}{\partial h^{(a)}_{\ \ \ l,\,k}}\, h^{(a)}_{\ \ \ l,\,i} - \delta_i^{\ k}\, \mathfrak{L}\right]. \end{aligned}\right\} \tag{2.20}$$

In terms of the tetrad fields, Einstein's field equations take the form

$$-\frac{1}{\varkappa}\mathfrak{G}_i^{\ k} \equiv \frac{1}{2\varkappa}\, h^{(a)}_{\ \ \ i}\, \frac{\delta \mathfrak{L}}{\delta h_k^{(a)}} \equiv \frac{1}{2\varkappa}\, h^{(a)\,k}\, \frac{\delta \mathfrak{L}}{\delta h^{(a)\,i}} = \mathfrak{T}_i^{\ k}, \tag{2.21}$$

where $\dfrac{\delta \mathfrak{L}}{\delta h_k^{(a)}}$ is the variational derivative of $\mathfrak{L}$ with respect to $h_k^{(a)}$. The complex $\mathsf{T}_i^{\ k}$ is derivable from a superpotential $\mathfrak{U}_i^{\ kl}$, i.e.

$$\mathsf{T}_i^{\ k} = \mathfrak{U}_i^{\ kl}{}_{,\,l} \tag{2.22}$$

with

$$\mathfrak{U}_i^{\ kl} = -\mathfrak{U}_i^{\ lk} = \frac{1}{2\varkappa}\frac{\partial \mathfrak{L}}{h^{(a)\,i}_{\ \ \ ,\,l}}\, h^{(a)\,k} = \frac{1}{2\varkappa}\, h_i^{(a)}\, \frac{\partial \mathfrak{L}}{\partial h^{(a)}_{l,\,k}} \tag{2.23}$$

(see Eqs. (2.31)–(2.38) in [2], and also [10]).

The explicit expressions for the complex $\mathfrak{t}_i^{\ k}$ and the superpotential are (Eqs. (2.39), (2.40) in [2])

$$\mathfrak{t}_i^{\ k} = \frac{|h|}{\varkappa}\left[h^{(a)\,k}{}_{;\,l}\, h_{(a),\,i}^{\ \ \ \ \ \ l} - h^{(a)\,r}{}_{;\,r}\, h_{(a),\,i}^{\ \ \ \ \ \ k} + h_r^{(a)}\, h_{(a),\,i}^{r}\, h^{(b)\,k}\, h_{(b);\,s}^{s}\right] - \frac{1}{2\varkappa}\,\delta_i^{\ k}\, \mathfrak{L} \tag{2.24}$$

$$\mathfrak{U}_i^{\ kl} = \frac{|h|}{\varkappa}\left[h_{(a)}^{k}\, h^{(a)\,l}{}_{;\,i} + (\delta_k^i\, h^{(a)\,l} - \delta_i^l\, h^{(a)\,k})\, h_{(a);\,s}^{s}\right]. \tag{2.25}$$

In contrast to the superpotential $h_i^{\ kl}$ in Einstein's theory, the superpotential $\mathfrak{U}_i^{\ kl}$ is seen to be a true tensor density of rank 3.

It is convenient to introduce the tensors

$$\left.\begin{aligned} \gamma_{ikl} &= h^{(a)}{}_i\, h_{(a)\,k;\,l} = -\gamma_{kil} \\ \Lambda_{ikl} &= \gamma_{ikl} - \gamma_{ilk} = h^{(a)}{}_i\,(h_{(a)\,k,\,l} - h_{(a)\,k,\,l}) = -\Lambda_{ilk} \\ \Phi_k &= \gamma^i{}_{ki} = \Lambda^i{}_{ki} = h^{(a)\,i}\, h_{(a)\,k;\,i} \end{aligned}\right\} \tag{2.26}$$

and the symbol

$$\Delta^i{}_{kl} = h^{(a)\,i}\, h_{(a)\,k,\,l} \tag{2.27}$$

which is connected with the Christoffel symbol by the relation

$$\Delta^i{}_{kl} = \Gamma^i_{kl} + \gamma^i{}_{kl}. \tag{2.28}$$

In terms of these quantities the expressions (2.19)–(2.25) take the form (Eqs. (3.12), (3.14), (D 34) in [(2)])

$$\mathfrak{L} = |\, h\,|\,[\gamma_{rst}\,\gamma^{tsr} - \Phi_r\,\Phi^r] \tag{2.29}$$

$$\mathfrak{U}_i^{kl} = \frac{|\,h\,|}{\varkappa}\,[\gamma^{kl}{}_i - \delta_i^k\,\Phi^l + \delta_i^l\,\Phi^k] \tag{2.30}$$

$$\mathfrak{t}_i^k = \frac{|\,h\,|}{\varkappa}\,[\gamma^{km}{}_l\,\Delta^l{}_{mi} - \Phi^l\,\gamma^k{}_{li} + \Delta^l{}_{li}\,\Phi^k] - \frac{1}{2\,\varkappa}\,\delta_i^k\,\mathfrak{L} \tag{2.31}$$

or

$$\mathfrak{t}_i^k = \mathfrak{U}_i^k + \mathfrak{U}_m{}^{kl}\,\Delta^m{}_{il}, \tag{2.32}$$

where $\mathfrak{U}_i^k$ is the tensor density

$$\mathfrak{U}_i^k = \frac{|\,h\,|}{\varkappa}\,\Lambda_{mli}\,\gamma^{klm} - \Phi_i\,\Phi^k + \Lambda^k{}_{il}\,\Phi^l] - \frac{1}{2\,\varkappa}\,\delta_i^k\,\mathfrak{L}. \tag{2.33}$$

All the quantities introduced here are true tensors or tensor densities, except $\mathfrak{t}_i^k$ and $\mathfrak{T}_i^k$ which deviate from a true tensor density by the term $\mathfrak{U}_m{}^{kl}\,\Delta^m_{il}$. Thus, we get the following transformation law for $\mathfrak{T}_i^k$ and $\mathfrak{t}_i^k$ under an arbitrary space-time transformation $(x^i) \to (x'^i)$ (Eq. (D. 37) in [(2)]):

$$\mathfrak{T}'^{\,k}_i = J\,\frac{\partial x'^k}{\partial x^m}\left[\frac{\partial x^l}{\partial x'^i}\,\mathfrak{T}_l^m + \frac{\partial}{\partial x^n}\left(\frac{\partial x^r}{\partial x'^i}\right)\mathfrak{U}_r{}^{mn}\right], \tag{2.34}$$

where J is the Jacobian of the transformation

$$J = \frac{\partial\,(x^1 \cdot \cdot\, x^4)}{\partial\,(x'^1 \cdot \cdot\, x'^4)} = \det\left\{\frac{\partial x^i}{\partial x'^k}\right\}. \tag{2.35}$$

From (2.34) we see that the most general group of transformations under which the quantities $\mathsf{T}^k \equiv \mathsf{T}_4{}^k$ transform as a 4-vector density are those for which

$$\frac{\partial x^r}{\partial x'^4} = \delta^r_4 . \tag{2.36}$$

This also follows from (2.22) because $\mathfrak{U}_i^{kl}$ is a true tensor density, i.e.

$$\mathfrak{U}'^{kl}_4 = J \frac{\partial x^r}{\partial x'^4} \frac{\partial x'^k}{\partial x^s} \frac{\partial x'^l}{\partial x^t} \mathfrak{U}_r{}^{st} = J \frac{\partial x'^k}{\partial x^s} \frac{\partial x'^l}{\partial x^t} \mathfrak{U}_4{}^{st} \tag{2.37}$$

on account of (2.36). Thus, $\mathfrak{U}_4{}^{kl}$ is an antisymmetric tensor density of rank 2 under all transformations of the type (2.36), which means that

$$\mathsf{T}^k = \mathfrak{U}_4{}^{kl}{}_{,l} \tag{2.38}$$

is a vector density.

The most general transformation of the type (2.36) has the form

$$\left.\begin{aligned} x'^{\iota} &= f^{\iota}(x^{\varkappa}) \\ x'^4 &= x^4 + f(x^{\varkappa}), \end{aligned}\right\} \tag{2.39}$$

where f^{ι} and f are arbitrary functions of the spatial coordinates. It contains the group of purely spatial transformations (2.5) as a subgroup. Under the latter group the fourth component of T^k, i.e. $\mathsf{T}_4{}^4$, is a scalar density, which means that

$$H_V = -\iiint\limits_V \mathsf{T}_4{}^4 \, dx^1 \, dx^2 \, dx^3 \tag{2.40}$$

is invariant.

If we now introduce the expression $\mathsf{T}_4{}^4 = \mathfrak{U}_4{}^{4\lambda}{}_{,\lambda}$ into (2.40) we get, by means of Gauss' theorem,

$$H_V = -\iint\limits_F \mathfrak{U}_4{}^{4\lambda} \, dS_\lambda , \tag{2.41}$$

where the integration is extended over the boundary surface F of the volume V. Here,

$$dS_\lambda = \delta_{\lambda\varkappa\mu} \, dx^{\varkappa} \, \delta x^{\mu}, \tag{2.42}$$

where $\delta_{\lambda\varkappa\mu}$ is the 3-dimensional Levi-Civita symbol and $dx^{\varkappa}$, δx^{μ} are infinitesimal 3-vectors spanning the surface element on F. Since $\mathfrak{U}_4{}^{4\lambda}$ transforms as a 3-vector density under the group of purely spatial transformations (2.5), $\mathfrak{U}_4^{4\lambda}\, dS_\lambda$ is invariant, which again shows that H_V in (2.41) is invariant under these transformations. The expression (2.41) is valid also if the spatial system of coordinates is composed of different coordinate patches.

Let us now consider the integrals

$$(P_V)_\iota = \iiint\limits_V \mathfrak{T}_\iota^4\, dx^1\, dx^2\, dx^3 = \iiint\limits_V \mathfrak{U}_\iota^{4\lambda}{}_{,\lambda}\, dx^1\, dx^2\, dx^3 \tag{2.43}$$

obtained from (2.4) with $i = \iota$ by integrating over a finite part V of 3-space. Then, if $\mathfrak{U}_\iota^{4\lambda}$ is continuous inside V, we get again by means of Gauss' theorem

$$(P_V)_\iota = \iint\limits_F \mathfrak{U}_\iota^{4\lambda}\, dS_\lambda = \iint\limits_F A_\iota(x). \tag{2.44}$$

Since $\mathfrak{U}_i^{kl}$ is a true tensor density, it follows that

$$A_\iota(x) \equiv \mathfrak{U}_\iota^{4\lambda}\, dS_\lambda \tag{2.45}$$

at each point on F transforms as a 3-vector under the group of spatial transformations (2.5). Nevertheless, the sum (or integral) of the components of the vectors A_ι in different points of F, of course, has in general no simple physical meaning. However, this should not be regarded as a defect of the theory, since we have a similar situation already in special relativity if we use curvilinear coordinates in space.

In the limit of special relativity where space-time can be regarded as flat, we have, in a system of inertia a momentum density given by the components $\mathfrak{T}_\iota^4$ of the matter tensor density, which transform as a 3-vector density under arbitrary spatial transformations, so that

$$B_\iota = \mathfrak{T}_\iota^4\, dx^1\, dx^2\, dx^3 \tag{2.46}$$

are the covariant components of a 3-vector. Nevertheless, the three integrals

$$(P_V^{(m)})_\iota = \iiint\limits_V \mathfrak{T}_\iota^4(x)\, dx^1\, dx^2\, dx^3 = \iiint\limits_V B_\iota(x) \tag{2.47}$$

have in general no physical meaning at all.

This will be the case only in a system of rectilinear coordinates where the integrals (2.47) are the components of the total linear momentum inside V, which is a free vector $\vec{P}_V^{(m)}$. Now, a free vector can by parallel displacement be attached to any point p in space, and in a system of rectilinear coordinates its components are the same in every point. However, by a transformation to curvilinear coordinates, the components of the free vector $\vec{P}_V^{(m)}$ will be different in different points p.

Of course, this does not prevent us from using curvilinear coordinates, but then we have in (2.47) to substitute the arithmetical sum of the vector components B_ι of the vector $\vec{B}$ by the geometrical sum of the vectors $\vec{B}(x)$. Thus, the components of the free vector $\vec{P}_V^{(m)}$ in a point p are obtained by parallel displacement of the vectors $\vec{B}(x)$ to the point p, i.e.

$$(P_V^{(m)})_\iota = \iiint_V B_\iota^*(p), \tag{2.48}$$

where the $B_\iota^*(p)$ are the components of the vectors obtained by parallel displacement of the vectors $B_\iota(x)$ from the various points (x) to the point p.

In special relativity, this procedure leads to a unique result in any system of coordinates, since the space is flat and the result of a parallel displacement therefore is independent of the curve along which the displacement has been made. However, in general relativity where the space may be curved, it would seem impossible in this way to get an unambiguous expression for the linear momentum of a physical system in a given system of reference. This is certainly also true if we consider the matter alone. It is different, however, if we consider the momentum of the complete system of matter plus gravitational field, in which case it turns out to be possible to get a unique expression for the total linear momentum at least for any *insular* system where the matter is confined to a finite part of space.

The reason for this is the following. For an insular system, space-time can be regarded as flat at sufficiently large spatial distances from the system and, consequently, we may introduce coordinates which are at least asymptotically rectilinear. Further, in contrast to the integrals (2.47), the quantities $(P_V)_\iota$ in (2.44) have the form of a sum of vector components A_ι situated on the boundary surface F. Therefore, if we make the volume V so large that F lies entirely in the region where the space may be regarded as flat, the situation is exactly as in special relativity. In a system of coordinates which is rectilinear in this region, the components of the total momentum of the physical system inside the surface F are consistently given by (2.44), i.e.

by the arithmetical sum of the components of the vectors A_ι on F. The total momentum $\vec{P}_V$ is a free vector and, since the quantities A_ι in (2.44) are 3-vectors under spatial transformations (2.5), we get the components $(P_V)_\iota(p)$ in a point p in arbitrary spatial curvilinear coordinates by parallel displacement of the vectors $\vec{A}(x)$ to the point p, i.e.

$$(P_V)_\iota(p) = \iint_F A_\iota^*(p) \tag{2.49}$$

on the analogy of (2.48). This gives a unique result provided that the curves along which the displacements are made are chosen to lie entirely inside the region where the space may be considered as flat.[1]

If we had used Einstein's expression $\Theta_i^{\ k}$ for the energy-momentum complex instead of $\mathsf{T}_i^{\ k}$, the just mentioned procedure would not have given consistent results for, in this case, the equations (2.43), (2.44) would be replaced by

$$(P_V^E)_\iota = \iiint_V \Theta_\iota^{\ 4}\, dx^1\, dx^2\, dx^3 = \iint_F h_\iota^{\ 4\lambda}\, dS_\lambda \tag{2.50}$$

and, in contrast to A_ι in (2.44), (2.45), the quantity $h_\iota^{\ 4\lambda}\, dS_\lambda$ does not transform as a 3-vector under the transformations (2.5) except if the functions $f^\iota(x^\varkappa)$ are linear. In curvilinear coordinates we would therefore not know how to perform the above mentioned parallel displacement, and we can only hope that the equations (2.50) give correct results for systems of coordinates which are asymptotically rectilinear. The results obtained in sections 4 and 5 of the present paper seem to justify this hope.

Finally, it should also be remarked that the preceding considerations are somewhat loose, since we have assumed that the space is flat for a sufficiently large surface F. Actually, the flatness of the space at large spatial distances from an insular system is only an asymptotic property and we have in each case to state more precisely how large the surface F has to be chosen.

There is another important question which we have disregarded so far. For a given tetrad field, the metric is determined by the equations (2.15), but for a given metric tensor $g_{ik}(x)$ the tetrad field is not uniquely deter-

[1] In a forthcoming paper it will be shown that the preceding considerations regarding the 3-momentum vector in curvilinear coordinates inside a given system of reference can easily be carried through also for the 4-momentum P_i in the 4-dimensional space, which enables us to give a meaning to P_i for arbitrary curvilinear space-time coordinates.

mined by these equations. If $h_i^{(a)}(x)$ represents a solution, then the tetrads

$$\check{h}_i^{(a)} = \Omega^{(a)}{}_{(b)}(x)\, h_i^{(b)} \tag{2.51}$$

also satisfy the equations (2.15), providedth at the scalar functions $\Omega^{(a)}{}_{(b)}(x)$ at each point (x) satisfy the orthogonality relations of a Lorentz rotation, i.e.

$$\Omega^{(a)}{}_{(c)}\, \Omega_{(b)}{}^{(c)} = \Omega_{(c)}{}^{(a)}\, \Omega^{(c)}{}_{(b)} = \delta_b^a \tag{2.52}$$

(the indices in parenthesis are lowered and raised by the same rule as in (2.14)).

It is usually assumed that all measurable physical quantities and all relations between such quantities must be invariant under arbitrary Lorentz rotations (2.51) of the tetrads. In the case of the covariant Dirac equation for fermion fields, for instance, measurable quantities such as the charge and current densities are unchanged under the transformations (2.51), in contrast to the field function $\psi(x)$ which transforms as an 'undor'. Now, the components of the complex $\mathsf{T}_i^{\ k}$ are invariant under (2.51) only if the rotation coefficients are constants $\overset{(o)}{\Omega}{}^{(a)}{}_{(b)}$, i.e. independent of (x). In fact, from the definition (2.26) of γ_{ikl} one finds at once the transformation law

$$\check{\gamma}^{ikl} \equiv \check{h}_i^{(a)}\, \check{h}_{(a)\,k;\,l} = \gamma_{ikl} + X_{ikl}, \tag{2.53}$$

where the tensor $X_{ikl} = -X_{kil}$ is given by

$$X_{ikl} = \Omega^{(c)}{}_{(a)}\, \Omega_{(cb),\,l}\, h_i^{(a)}\, h_k^{(b)}. \tag{2.54}$$

Further, since $h = \sqrt{-g}$ is invariant under (2.51), the transformation of the superpotential (2.30) is given by

$$\check{\mathfrak{U}}_i^{\ kl} = \mathfrak{U}_i^{\ kl} + \mathrm{Y}_i^{\ kl} \tag{2.55}$$

with

$$\mathrm{Y}_i^{\ kl} = -\mathrm{Y}_i^{\ lk} = \frac{|h|}{\varkappa}\left[X^{kl}{}_i - \delta_i^k X^l + \delta_i^l X^k\right] \tag{2.56}$$

$$X^k = X^{ik}{}_i = \Omega^{(c)}{}_{(a)}\, \Omega_{(cb),\,i}\, h^{(a)\,i}\, h^{(b)\,k}. \tag{2.57}$$

Finally, we get from (2.22) the following transformation equation for the complex $\mathsf{T}_i^{\ k}$ under tetrad rotations (2.51):

$$\check{\mathsf{T}}_i^{\ k} = \check{\mathfrak{U}}_i^{\ kl}{}_{,\,l} = \mathsf{T}_i^{\ k} + \mathrm{Y}_i^{\ kl}{}_{,\,l} \tag{2.58}$$

and the last term is in general not zero, unless the rotation coefficients $\Omega^{(a)}{}_{(b)}$ are constant.

Thus, Einstein's field equations which determine the metric only do not allow to calculate the complex $\mathsf{T}_i^{\ k}$ uniquely. Therefore, if one regards the energy density or more generally $\mathsf{T}_i^{\ k}$ as a measurable quantity, one will try to set up further equations which, together with Einstein's field equations, allow to determine the tetrad field so accurately that $\mathsf{T}_i^{\ k}$ can be calculated uniquely. Geometrically speaking, besides the curvature of 4-space which is determined by Einstein's equations, we need a set of supplementary equations which allow to calculate the torsion of the space (or the tetrad lattice), i.e. the tensor γ_{ikl}.

In the trivial case of a completely empty space where $\mathfrak{T}_i^{\ k}$ is everywhere zero, one usually assumes that space-time is flat, i.e.

$$R_{iklm} = 0. \tag{2.59}$$

In that case we must assume that also the torsion is zero, i.e.

$$\gamma_{ikl} = 0 \quad \text{or} \quad h^{(a)}_{i;\,k} = 0, \tag{2.60}$$

for only with this assumption will $\mathsf{T}_i^{\ k}$ be equal to zero, as we should have for a completely empty space. Since R_{iklm} is a linear function of the covariant derivatives of the $h_{(a)\,i}$ of the second order, the equations (2.60) are compatible with (2.59). In contrast to $\Theta_i^{\ k}$ which is different from zero in curvilinear coordinates, the covariant equations (2.60) ensure that $\mathsf{T}_i^{\ k} = 0$ in all systems of coordinates.

The equations (2.60) can also be written

$$\gamma_{(abc)} = 0, \tag{2.61}$$

where

$$\gamma_{(abc)} = \gamma_{ikl}\, h^i_{(a)}\, h^k_{(b)}\, h^l_{(c)} = h_{(a)\,k;\,l}\, h^k_{(b)}\, h^l_{(c)} \tag{2.62}$$

are the Ricci rotation coefficients. This means that the tetrad field in a completely empty space has to be chosen so that 'absolute' parallelism with respect to these tetrads (see reference 2, section 5) coincides with the Levi-Civita parallelism which is also global in the case of a flat space, where we can use pseudo-Cartesian or Lorentzian coordinates. In such coordinates

$$g_{ik} = \eta_{ik} \tag{2.63}$$

and (2.60), (2.61) mean that we may choose

$$h_i^{(a)} = \delta_i^a \tag{2.64}$$

or, more generally,

$$h_i^{(a)} = \overset{(o)}{\Omega}{}^{(a)}{}_{(i)}, \tag{2.65}$$

where $\overset{(o)}{\Omega}{}^{(a)}{}_{(b)}$ is any set of constant rotation coefficients satisfying the relations (2.52).

For a system with matter it is clear that (2.60) or (2.61) cannot be valid, for this would entail (2.59) which would be in contradiction with Einstein's field equations. However, for an insular system where space-time is asymptotically flat at large spatial distances r, (2.61) must hold asymptotically, i.e.

$$\gamma_{(abc)} \to 0 \quad \text{for} \quad r \to \infty. \tag{2.66}$$

Thus, in an asymptotically Lorentzian system of coordinates the tetrads must satisfy the boundary conditions.

A. $$h_i^{(a)} - \overset{(o)}{\Omega}{}^{(a)}{}_{(i)} \to \infty \quad \text{for} \quad r \to \infty. \tag{2.67}$$

Further, as regards the manner in which this quantity tends to zero, we shall make the following natural assumption:

B. $h_i^{(a)} - \overset{(o)}{\Omega}{}^{(a)}{}_{(i)}$ *shows the same asymptotic behaviour as the metric quantities* $g_{ik} - \eta_{ik}$.

This behaviour depends of course on the type of physical system we are dealing with. For a system with outgoing radiation, only, the boundary condition will have the character of Sommerfeld's radiation condition. The form of the boundary conditions will of course also depend on the system of coordinates. Although it may be convenient to use asymptotically rectilinear coordinates, the boundary conditions can of course be formulated in any system of coordinates.

As regards the supplementary equations which, together with Einstein's field equations and the boundary conditions, should determine the complex $\mathsf{T}_i^{\ k}$ uniquely, it was shown in reference 2 that the following six covariant equations would serve this purpose:

$$\gamma_{ik}{}^{l}{}_{;l} + \gamma_{ikl}\,\Phi^l = 0. \tag{2.68}$$

However, these equations are not the only possible ones and the arbitrariness in the choice of the supplementary equations is even somewhat larger than was assumed in reference 2. In an interesting paper by Pellegrini and Plebansky[11], this arbitrariness is diminished by the requirement that all the equations for the tetrad field should be derived from a variational principle. In this way, they arrive at a theory which, in the weak-field approximation and in the case of a spherically symmetric system, is practically identical with the theory in reference 2. But in addition to that, their formulation opens up the possibility of an interesting generalization of the usual theory of gravitation, in particular as regards systems containing fermion fields. However, also this theory contains some arbitrariness and the equations determining the metric tensor are not exactly identical with Einstein's field equations. For this reason, it may be for the time preferable to work with the formulation developed in reference 2.

When we have made a certain choice as regards the supplementary equations, for instance the equations (2.68), the energy-momentum complex for a given physical system is a definite function of the space-time coordinates, which means that we may calculate the energy distribution throughout space. However, if it is true that the energy content in a small part of space is unmeasurable, then we have obviously obtained too much. Now, it is an interesting fact that, if we only regard the *total* energy and momentum as measurable quantities, the question of the exact form of the supplementary conditions does not arise. In fact, as we shall see in section 6, the energy and momentum contained in a sufficiently large volume V are invariant under all tetrad rotations (2.51) which respect the boundary conditions A and B for the tetrads formulated on p. 18. On the other hand, the distribution of the energy throughout space will in general be different after a tetrad rotation. This is quite satisfactory if the energy *distribution* is unmeasurable. The situation is then here somewhat similar to the case of the covariant Dirac equation where the measurable quantities, like the charge and current densities, are invariant under tetrad rotations, while the wave functions themselves are not invariant. From this point of view the tetrad field variables have to be regarded as subsidiary quantities like the potentials in electrodynamics, and the tetrad rotations are a kind of gauge transformations under which the measurable quantities, such as total energy and moment, are invariant. Supplementary equations of the type (2.68) are then not necessary, but sometimes it may be convenient to 'fix the gauge' by applying such covariant equations.*

3. The Gravitational Field at Large Spatial Distances from an Insular System with Axial Symmetry

In order to calculate the gravitational energy emitted from a physical system as well as the total energy and momentum of the system by means

* In some cases it may even be advantageous, just as in electrodynamics, to fix the gauge in a non-covariant way; this is not in contradiction with the principle of relativity, since the gauge of the tetrads in this point of view is considered unobservable. In this connection, cf. also reference 14.

of the theory outlined in the preceding section, we have to know the field at large spatial distances only. In reference 1, Bondi, van der Burg and Metzner have given the exact form of the metric at large spatial distances from any axi-symmetric system with no ingoing radiation. These investigations were extended to an arbitrary system in a subsequent paper by Sachs[(12)]. In the present paper we shall, for simplicity, confine ourselves to the consideration of axi-symmetric systems and start by quoting some of the relevant results obtained by Bondi et al.

Although it is in principle allowed to use any system of coordinates in general relativity, there are certain classes of coordinate systems in which the boundary conditions have a particularly simple form.

In the system of coordinates S' with coordinates

$$(x'^i) = \{r, \theta, \varphi, u\} \tag{3.1}$$

introduced by Bondi et al., θ and φ are a kind of polar angles with the symmetry axis as polar axis, and r is a 'radial' coordinate chosen in such a way that the 2-surface $du = dr = 0$ has the area $4\pi r^2$. Further, the time variable u is defined so that the curve $du = d\theta = d\varphi = 0$ represents an *outgoing* light ray.

In S' the metric tensor g'_{ik} has the form

$$g'_{ik} = \begin{pmatrix} 0 & 0 & 0 & -e^{2\beta} \\ 0 & r^2 e^{2\gamma} & 0 & -r^2 U e^{2\gamma} \\ 0 & 0 & r^2 \sin^2\Theta\, e^{-2\gamma} & 0 \\ -e^{2\beta} & -r^2 U e^{2\gamma} & 0 & -(r^{-1} V e^{2\beta} - r^2 U^2 e^{2\gamma}) \end{pmatrix} \tag{3.2}$$

where U, V, β, γ are functions of r, θ and u. The corresponding determinant $g' = \det\{g'_{ik}\}$ is given by

$$\sqrt{-g'} = r^2 \sin\theta\, e^{2\beta}. \tag{3.3}$$

The contravariant components of the metric tensor are

$$g'^{ik} = \begin{pmatrix} r^{-1} V e^{-2\beta} & -U e^{-2\beta} & 0 & -e^{-2\beta} \\ -U e^{-2\beta} & r^{-2} e^{-2\gamma} & 0 & 0 \\ 0 & 0 & (r\sin\theta)^{-2} e^{2\gamma} & 0 \\ -e^{-2\beta} & 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

To ensure regularity in the neighbourhood of the polar axis the functions V, β, $U/\sin\theta$, $\gamma/\sin^2\theta$ have to be regular as $\sin\theta$ goes to zero. Although different coordinate patches in general may be necessary throughout space-time, it is believed that the space sufficiently far from the system is covered by one patch of coordinates of the type (3.1)–(3.4). In these coordinates the absence of inward flowing radiation may be expressed by the assumption that the functions U, V, β, γ for sufficiently large distances r can be written as a power series in $1/r$ with coefficients depending on θ and u only. By introduction of the corresponding series expansion for g'_{ik} into Einstein's field equations for the empty space outside the matter one obtains

$$\left.\begin{aligned} \gamma &= c(u,\theta)\, r^{-1} + O_3 \\ \beta &= -\frac{1}{4} c(u,\theta)^2\, r^{-2} + O_3 \\ U &= -(c + 2\, c \cot\theta)\, r^{-2} \\ &\quad + [2\, N(u,\theta) + 3\, cc_2 + 4\, c^2 \cot\theta]\, r^{-3} + O_4 \\ V &= r - 2\, M(u,\theta) \\ &\quad - \left[N_2 + N \cot\theta - c_2{}^2 - 4\, cc_2 \cot\theta - \frac{1}{2} c^2 (1 + 8 \cot^2\theta)\right] r^{-1} + O_2 . \end{aligned}\right\} \tag{3.5}$$

Here, O_n means a term which vanishes as r^{-n} for $r \to \infty$. $c(u, \Theta)$, $M(u, \Theta)$, $N(u, \Theta)$ are functions of integrations which depend on the type of matter system we are dealing with and the suffix 2 means partial differentiation with respect to Θ for constant r, φ, u. In general we shall use the notation

$$(\ )_1 = \frac{\partial}{\partial r}, (\ )_2 = \frac{\partial}{\partial \theta}, (\ )_3 = \frac{\partial}{\partial \varphi}, (\ )_0 = \frac{\partial}{\partial u}. \tag{3.6}$$

The functions c, M, N are not independent, they are connected by the relations

$$\left.\begin{aligned} M_0 &= -c_0{}^2 + \frac{1}{2} A_0 \\ -3\, N_0 &= M_2 + 3\, cc_{02} + 4\, cc_0 \cot\theta + c_0\, c_2 \end{aligned}\right\} \tag{3.7}$$

with

$$\left.\begin{aligned} A &= c_{22} + 3\, c_2 \cot\theta - 2\, c \\ &= (c_2 \sin\theta + 2\, c \cos\theta)_2 / \sin\theta \\ &= \left[\frac{(c \sin^2\theta)_2}{\sin\theta}\right]_2 / \sin\theta . \end{aligned}\right\} \tag{3.8}$$

Thus, if $c(u, \theta)$ is given, the functions M and N may be obtained by integration of (3.7) and the flow of information in the system is entirely controlled by the function c which, in reference 1, is called the *news function*. Further, since

$$g'_{44} = -\left(1 - \frac{2\,M(u,\theta)}{r} + O_2\right) \tag{3.9}$$

on account of (3.2) and (3.5), the function $M(u, \theta)$ is called the *mass aspect*. In the case of a static system, M is simply equal to the total mass m of the system, i.e.

$$M = m. \tag{3.10}$$

In part D of reference 1, Bondi proves the interesting theorem that the mean value $m(u)$ of $M(u, \Theta)$ over all directions is a never increasing function of time. In fact, with

$$m(u) = \frac{1}{2}\int_0^\pi M(u,\theta)\sin\theta\, d\theta \tag{3.11}$$

we get, by means of (3.7), (3.8),

$$-m(u)_0 = -\frac{1}{2}\int_0^\pi M_0 \sin\theta\, d\theta = \frac{1}{2}\int_0^\pi c_0^{\,2}\sin\theta\, d\theta. \tag{3.12}$$

Here we have used that

$$\int_0^\pi A\sin\theta\, d\theta = \left.\frac{(c\sin^2\theta)_2}{\sin\theta}\right|_0^\pi = 0 \tag{3.13}$$

on account of the regularity condition for $\gamma/\sin^2\theta$ for $\sin\theta \to 0$ which, by the first equation (3.5), leads to the following limiting behaviour for the news function c:

$$c \simeq k(u)\sin^2\theta \quad \text{for} \quad \sin\theta \to 0. \tag{3.14}$$

For a static system the quantity (3.11) is equal to the total mass or energy of the system and, since the right-hand side in (3.12) is always positive unless $c_0 = 0$, we see that a system which is initially and finally static must lose energy if the news function c_0 is different from zero in the intermediate stage.

This important result of BONDI was limited to the case where the system initially and finally is in a static state. By means of the theory of the preceding section we are now able to prove this theorem for any (axi-symmetric) system and we shall also verify Bondi's conjecture that the quantity $m(u)$ in (3.11) is equal to the total energy for all times, so that the formula (3.12) gives the energy radiated per unit time at any stage of the development of the system.

Let us first calculate the energy flux through a large sphere of 'radius' r using Einstein's energy-momentum complex $\Theta_i^{\,k}$. According to (2.10) and (3.3), the energy current density should be

$$-\vartheta_4'^{\varkappa} = -\frac{r^2 \sin\Theta}{2\varkappa}\left[\Gamma'^{\varkappa}_{lm}\,(e^{2\beta} g'^{lm})_0 - \Gamma'^{s}_{ms}\,(e^{2\beta} g^{\varkappa m})_0\right]. \tag{3.15}$$

Further, if

$$dx'^{\varkappa} = \{0,\, d\theta,\, 0\},\quad \delta x'^{\varkappa} = \{0,\, 0,\, d\varphi\} \tag{3.16}$$

are two infinitesimal 3-vectors lying on the sphere of radius r in the directions of increasing θ and φ, respectively, the quantity (2.42), representing the surface element spanned by these vectors, becomes

$$dS'_{\varkappa} = \delta_{\varkappa\lambda\mu}\, dx'^{\lambda}\,\delta x'^{\mu} = \{d\theta\, d\varphi,\, 0,\, 0\}. \tag{3.17}$$

The energy flux in the outward direction through this surface element should then be

$$S'_E\, d\theta\, d\varphi = -\vartheta_4'^{\mu}\, dS'_{\varkappa} = -\vartheta_4'^{1}\, d\theta\, d\varphi. \tag{3.18}$$

By introducing the series expansion of the metric tensor (3.2)–(3.3) following from (3.5) into the right-hand side of (3.15) we get, as shown in the Appendix,

$$-\vartheta_4'^{\varkappa} = \frac{2\sin\theta}{\varkappa}\left[c_0^{\,2} - M_0 + \frac{1}{4}\cot\theta\,(c_{20} + 2\,c_0\cot\theta)\right]\delta_1^{\varkappa} + O_1. \tag{3.19}$$

For sufficiently large values of r we can neglect the term O_1 and we get for the differential energy flux (3.18) by means of the first equation (3.7)

$$S'_E\, d\theta\, d\varphi = \frac{2}{\varkappa}\left[2\,c_0^{\,2} - \frac{1}{2}A_0 + \frac{1}{4}\cot\theta\,(c_{20} + 2\,c_0\cot\theta)\right]\sin\theta\, d\theta\, d\varphi. \tag{3.20}$$

Thus, according to Einstein's expression for the energy flux, the total energy which per unit time is leaving a sphere with a sufficiently large radius should be

$$\iint S'_E \, d\theta \, d\varphi = \int_0^\pi c_0{}^2 \sin\theta \, d\theta + \frac{1}{8}\int_0^\pi c_0 \frac{1+\cos^2\theta}{1-\cos^2\theta} \sin\theta \, d\theta. \tag{3.21}$$

Here, we have introduced the value $\varkappa = 8\pi$ and used the equation (3.13) as well as the condition (3.14), which gives

$$\left.\begin{aligned} &\int_0^\pi \left(c_{02} \cos\theta + 2\, c_0 \frac{\cos^2\theta}{\sin\theta} \right) d\theta = c_0 \cos\theta \Big|_0^\pi \\ +&\int_0^\pi \left(2\, c_0 \frac{\cos^2\theta}{\sin\theta} + c_0 \sin\theta \right) d\theta = \int_0^\pi c_0 \frac{1+\cos^2\theta}{1-\cos^2\theta} \sin\theta \, d\theta. \end{aligned}\right\} \tag{3.22}$$

The expression (3.21) is not in accordance with Bondi's equation (3.12), in particular it does not have the essential property of being always positive since the integrant in the last term is linear in c_0.

The inadequacy of Einstein's expression Θ_i^k in the system of coordinates used by Bondi et al. is even more apparent if we calculate the total energy in a large sphere of radius r. By means of (2.13), (2.11) and (3.17) we get for this quantity

$$\left.\begin{aligned} H'_E(r) &= -\iiint \theta'^4_4 \, dx'^1 \, dx'^2 \, dx'^3 \\ &= -\iiint \frac{\partial h'^{4\lambda}_4}{\partial x'^\lambda} dx'^1 \, dx'^2 \, dx'^3 = -\iint h'^{4\lambda}_4 \, dS'_\lambda \\ &= -\iint h'^{41}_4 \, d\theta \, d\varphi, \end{aligned}\right\} \tag{3.23}$$

provided that the system of coordinates can be continued into the interior of the matter system in such a way that $h'^{4\lambda}_4$ is everywhere continuous. In the Appendix it is shown that h'^{41}_4 for large values of r is of the form

$$h'^{41}_4 = -\frac{2\, r \sin\theta}{\varkappa} + O_0, \tag{3.24}$$

which shows that

$$H'_E(r) \to \infty \quad \text{for} \quad r \to \infty. \tag{3.25}$$

It should be noted that the first term in (3.24), which causes the divergence, is completely independent of the functions c and M which characterize the system, i.e. the divergence is of the type mentioned earlier which was noticed already by BAUER(6) for a completely empty world. If we, quite arbitrarily, subtract this infinity, the term O_0 in (3.24), when introduced into (3.23), does not give the correct value of the total energy even for a static system. According to the considerations in the preceding section, this could also be expected.

4. Gravitational Energy Radiation from an Axi-Symmetric System

In this section we shall show that the complex $\mathsf{T}_i^{\,k}$ (in contrast to $\Theta_i^{\,k}$) gives a value for the energy radiation which is in agreement with Bondi's equation (3.12). In performing the calculations it is convenient to introduce a new system of coordinates S with coordinates

$$(x^i) = \{x, y, z, t\} \tag{4.1}$$

connected with the coordinates (3.1) of the system S' by the transformation

$$\left.\begin{aligned} &x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta \\ &x^4 = t = r + u. \end{aligned}\right\} \tag{4.2}$$

The advantage of the system S is that it is asymptotically Lorentzian and that the components of the metric tensor have a series expansion in $1/r$ starting with the power zero. Since $\mathsf{T}_i^{\,k}$ and $\mathfrak{U}_i^{\,kl}$ transform in a simple way, it is easy afterwards to find the components of these quantities in the system S'.

The transformation coefficients corresponding to (4.2) are

$$\left.\frac{\partial x^i}{\partial x'^k} = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi & 0 \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \\ \cos\theta & -r\sin\theta & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\right\} \tag{4.3}$$

with

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{4.4}$$

The corresponding Jacobian is

$$J = \det\left\{\frac{\partial x^i}{\partial x'^k}\right\} = r^2\sin\theta\,. \tag{4.5}$$

Therefore, by (3.3),

$$\sqrt{-g} = \frac{1}{J}\sqrt{-g'} = e^{2\beta}. \tag{4.6}$$

For the calculations it is convenient to introduce the following four-component quantities:

$$\left.\begin{aligned} n_i &= \frac{\partial r}{\partial x^i} = \{\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta,\ 0\} \\ m_i &= \{\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta,\ 0\} \\ l &= \{-\sin\varphi,\ \cos\varphi,\ 0,\ 0\} \\ \mu_i &= \frac{\partial u}{\partial x^i} = -(\eta_{i4}+n_i) = \{-\sin\theta\cos\varphi,\ -\sin\theta\sin\varphi,\ -\cos\theta,\ 1\} \end{aligned}\right\} \tag{4.7}$$

and the corresponding quantities n^i, m^i, l^i, μ^i with indices raised by means of the constant matrix $\eta^{ik} = \eta_{ik}$, i.e.

$$n^i = n_i,\ m^i = m_i,\ l^i = l_i,\ \mu^i = \eta^{ik}\mu_k = \{\mu_\iota,\ -1\}. \tag{4.8}$$

Then, obviously,

$$\left.\begin{aligned} & n_i n^i = m_i m^i = l_i l^i = 1, \qquad \mu_i \mu^i = 0 \\ & n_i m^i = n_i l^i = m_i l^i = m_i \mu^i = l_i \mu^i = 0,\ n_i \mu^i = -1. \end{aligned}\right\} \tag{4.9}$$

The derivatives of the quantities (4.7) with respect to u and r are zero, i.e.

$$(n_i)_0 = (n_i)_1 = (m_i)_0 = \cdots\cdots = (\mu_i)_0 = (\mu_i)_1 = 0 \tag{4.10}$$

and the derivatives with respect to θ and φ are at once seen to be

$$\left.\begin{aligned} & (n_i)_2 = m_i,\ (m_i)_2 = -n_i,\ (l_i)_2 = 0,\ (\mu_i)_2 = -m_i \\ & (n_i)_3 = \sin\theta\, l_i,\ (m_i)_3 = \cos\theta\, l_i, \\ & (l_i)_3 = -(\cos\theta\, m_i + \sin\theta\, n_i) \\ & (\mu_i)_3 = -\sin\theta\, l_i. \end{aligned}\right\} \tag{4.11}$$

Further, it follows from (4.7) that

$$\left.\begin{aligned} & m_i m_k + l_i l_k + \mu_i \mu_k + \eta_{i4}\mu_k + \mu_i \eta_{k4} \\ & = m_i m_k + l_i l_k - (\mu_i n_k + n_i \mu_k) - \mu_i \mu_k = \eta_{ik}. \end{aligned}\right\} \tag{4.12}$$

The transformation coefficients (4.3) may now be written as a row matrix

$$\left.\begin{aligned} &\frac{\partial x^i}{\partial x'^k} = (-\mu^i,\ rm^i,\ r\sin\theta\, l^i,\ \delta^i_4) \\ &\quad k \ = \quad 1 \qquad 2 \qquad\quad 3 \qquad 4. \end{aligned}\right\} \tag{4.13}$$

The inverse coefficients are then the column matrix

$$\frac{\partial x'^i}{\partial x^k} = \begin{pmatrix} n_k \\ r^{-1}\, m_k \\ (r\sin\theta)^{-1}\, l_k \\ \mu_k \end{pmatrix} \tag{4.14}$$

since

$$\left.\begin{aligned} \frac{\partial x^i}{\partial x'^l}\frac{\partial x'^l}{\partial x^k} &= -\mu^i n_k + m^i m_k + l^i l_k + \delta^i_4 \mu_k \\ &= m^i m_k + l^i l_k + \mu^i \mu_k + \mu^i \eta_{k4} + \eta^i_4 \mu_k = \delta^i_k \end{aligned}\right\} \tag{4.15}$$

on account of (4.12).

For the covariant components of the metric tensor in S we get, by (3.2) and (4.14),

$$\left.\begin{aligned} g_{ik} = \frac{\partial x'^l}{\partial x^i}\frac{\partial x'^m}{\partial x^k} g'_{lm} &= e^{2\gamma} m_i m_k + e^{-2\gamma} l_i l_k \\ -(r^{-1} V e^{2\beta} - r^2 U^2 e^{2\gamma})\, \mu_i \mu_k &- e^{2\beta}(\mu_i n_k + n_i \mu_k) \\ &- rUe^{2\gamma}(m_i \mu_k + \mu_i m_k). \end{aligned}\right\} \tag{4.16}$$

Similarly, by (3.4) and (4.13),

$$\left.\begin{aligned} g^{ik} = \frac{\partial x^i}{\partial x'^l}\frac{\partial x^k}{\partial x'^m} g'^{lm} &= e^{-2\gamma} m^i m^k + e^{2\gamma} l^i l^k + r^{-1} V e^{-2\beta} \mu^i \mu^k \\ + e^{-2\beta}(\mu^i \eta^k_4 + \eta^i_4 \mu^k) &+ rUe^{-2\beta}(\mu^i m^k + m^i \mu^k). \end{aligned}\right\} \tag{4.17}$$

At large distances r, the components of the metric tensor appear as a power series in $1/r$ with coefficients depending on u, θ, φ. In the following we shall only need explicit expressions for the terms up to the second power. By introduction of the expansions (3.5) into (4.16), and using (4.12), one easily finds

$$g_{ik} = \eta_{ik} + y_{ik} + z_{ik} + O_3 \tag{4.18}$$

with

$$y_{ik} = \alpha_{ik}\, r^{-1}, \qquad z_{ik} = \beta_{ik}\, r^{-2} \tag{4.19}$$

and

$$\left.\begin{aligned} \alpha_{ik} = \alpha_{ki} &= 2\,c\,(m_i\, m_k - l_i\, l_k) + 2\,M\,\mu_i\,\mu_k \\ &\quad + (c_2 + 2\,c\cot\theta)\,(m_i\,\mu_k + \mu_i\, m_k), \\ \beta_{ik} = \beta_{ki} &= 2\,c^2\,(m_i\, m_k + l_i\, l_k) + \frac{1}{2}\,c^2\,(\mu_i\, n_k + n_i\,\mu_k) \\ &\quad + (N_2 + N\cot\theta)\,\mu_i\,\mu_k - (2\,N + cc_2)\,(m_i\,\mu_k + \mu_i\, m_k). \end{aligned}\right\} \tag{4.20}$$

If we define the quantities y^{ik}, z^{ik}, α^{ik}, β^{ik} with indices raised by means of the constant matrix η^{ik} as in (4.8), then α^{ik} and β^{ik} are obtained from (4.20) by substituting m_i, l_i, μ_i and n_i by the quantities (4.8).

Now, it is easily seen that the contravariant components of the metric tensor have the following series expansion:

$$g^{ik} = \eta^{ik} - y^{ik} - z^{ik} + y^{ir}\, y_r{}^k + O_3, \tag{4.21}$$

for this expression satisfies the relation

$$g^{il}\, g_{kl} = \delta^i_k \tag{4.22}$$

up to the terms of second power in $1/r$. The expression (4.21) can also be obtained by introduction of the expansions (3.5) into (4.17).

Let $\psi\,(u, r, \theta, \varphi)$ be any function of the variables (x'^i) and let us denote the derivatives with respect to x^l by

$$\psi_{,\,l} \equiv \frac{\partial\,\psi}{\partial\, x^l} \tag{4.23}$$

with a comma in front of the index l, in contrast to the derivatives (3.6) with respect to x'^l which are written without a comma.

Then, by means of (4.14),

$$\left.\begin{aligned} \psi_{,\,l} &= \frac{\partial\, x'^m}{\partial\, x^l}\,\frac{\partial\,\psi}{\partial\, x'^m} \\ &= \psi_0\,\mu_l + \psi_1\, n_l + \frac{\psi_2\, m_l}{r} + \frac{\psi_3\, l_l}{r\sin\theta}. \end{aligned}\right\} \tag{4.24}$$

If ψ is equal to the function y_{ik} defined by (4.19), we have

$$y_{ik,\,l} = \frac{(\alpha_{ik})_0}{r}\,\mu_l + \frac{-\alpha_{ik}\,n_l + (\alpha_{ik})_2\,m_l + (\alpha_{ik})_3\,l_l/\sin\theta}{r^2}. \tag{4.25}$$

On the other hand, if ψ is a quantity of the type O_2, like z_{ik} or $y_{ir}\,y^r_k$, we have simply

$$z_{ik,\,l} = (z_{ik})_0\,\mu_l + O_3 = \frac{(\beta_{ik})_0}{r^2}\,\mu_l + O_3. \tag{4.26}$$

From (4.6), (3.5) and (4.21) we get the following expansion for $\mathfrak{g}^{ik} = \sqrt{-g}\,g^{ik}$

$$\mathfrak{g}^{ik} = \eta^{ik} - \frac{\alpha^{ik}}{r} - \frac{\beta^{ik} - \alpha^{ir}\,\alpha^k_r + \eta^{ik}\,c^2/2}{r^2} + O_3 \tag{4.27}$$

and, by means of (4.25), (4.26) and (4.9)–(4.11), one finds, as seen in the Appendix (A.1–A.8),

$$\mathfrak{g}^{ik}{}_{,\,k} = \frac{2\,M - A}{r^2}\,\mu^i + O_3. \tag{4.28}$$

This shows that our system of coordinates is harmonic only apart from terms of the type O_2.

In order to calculate the total energy and momentum as well as the energy emitted by means of the complex $\mathsf{T}_i^{\,k}$ we need an expression for the tetrad field $h^{(a)}{}_i$ corresponding to the metric (4.18)–(4.21). As explained in section 2, the tetrad field is not uniquely determined by the equation (2.15), since any tetrad rotation (2.51) will leave the metric unchanged. However, as was mentioned already in section 2 and as will be shown in detail in section 6, the values of the total energy and momentum given by (2.41) and (2.44) with a sufficiently large surface F are the same for any tetrad field satisfying (2.15) and the boundary conditions A and B on p. 18. Therefore we can choose any tetrad field satisfying these conditions, for instance

$$h_{(a)\,i} = \eta_{ai} + \frac{1}{2}\,y_{ai} + \frac{1}{2}\left(z_{ai} - \frac{1}{4}\,y_{ar}\,y^r_i\right) + O_3, \tag{4.29}$$

which is symmetrical in a and i. This expression is in accordance with the equation (2.15), for we have

$$h_i^{(a)}\, h_{(a)\,k} = \eta_{ik} + \frac{1}{2}(y_{ik} + y_{ki}) + \frac{1}{2}\left(z_{ik} - \frac{1}{4}\, y_{ir}\, y_k^r\right)$$
$$+ \frac{1}{2}\left(z_{ki} - \frac{1}{4}\, y_{kr}\, y_i^r\right) + \frac{1}{4}\, y_i^a\, y_{ak} + O_3$$
$$= \eta_{ik} + y_{ik} + z_{ik} + O_3 = g_{ik}$$

on account of (4.18). Further, since $h_{(a)\,i}$ in (4.29) is an algebraic function of the quantities y_{ik} and z_{ik} entering in the expression (4.18) for $g_{ik} - \eta_{ik}$, it is clear that this tetrad field satisfies the conditions A and B.

We shall now calculate the complex $\mathfrak{t}_i^k$, defined by (2.31), up to terms of power 2 in $1/r$. To this end, we only need to calculate the quantities (2.26), (2.27), which occur quadratically in $\mathfrak{t}_i^k$ up to the first power in $1/r$. According to (4.18), the Christoffel symbols are simply

$$\Gamma_{i,\,kl} = \frac{1}{2}(y_{ik,\,l} + y_{il,\,k} - y_{kl,\,i}) + O_2\,. \tag{4.30}$$

Further, since

$$\left.\begin{aligned} \gamma_{ikl} &= h_i^{(a)}\, h_{(a)\,k;\,l} = h_i^{(a)}\,(h_{(a)\,k,\,l} - \Gamma_{kl}^r\, h_{(a)\,r}) \\ &= h_i^{(a)}\, h_{(a)\,k,\,l} - \Gamma_{i,\,kl}, \end{aligned}\right\} \tag{4.31}$$

we get, by (4.29), (4.30) and (4.25),

$$\left.\begin{aligned} \gamma_{ikl} &= \delta_i^a\, \frac{1}{2}\, y_{ak,\,l} - \frac{1}{2}(y_{ik,\,l} + y_{il,\,k} - y_{kl,\,i}) + O_2 \\ &= \frac{1}{2}(y_{kl,\,i} - y_{il,\,k}) + O_2 \\ &= \frac{1}{2\,r}\,[(\alpha_{kl})_0\, \mu_i - (\alpha_{il})_0\, \mu_k] + O_2\,. \end{aligned}\right\} \tag{4.32}$$

Thus,

$$\Phi^k = \gamma^i{}_{ki} = \frac{1}{2\,r}\,[(\alpha_{ki})_0\, \mu^i - (\alpha_i^i)_0\, \mu_k] + O_2 = O_2 \tag{4.33}$$

on account of the relations

$$\alpha_i^i = 0, \quad \alpha_{ki}\, \mu^i = 0 \tag{4.34}$$

following from (4.20) and (4.9).

From (4.32)–(4.34) and (4.9) one easily finds that the Lagrangian (2.29) is zero up to terms of the second power in $1/r$. In fact, we have

$$\mathfrak{L} = \frac{1}{4\,r^2}\left[(\alpha_{kl})_0\,\mu_i - (\alpha_{il})_0\,\mu_k\right]\left[(\alpha^{ki})_0\,\mu^l - (\alpha^{li})_0\,\mu^k\right] + O_3 = O_3. \tag{4.35}$$

Finally, since

$$\left.\begin{aligned} \Delta^l{}_{mi} &= h^{(a)\,l}\,h_{(a)\,m,\,i} = \eta^{al}\,\frac{1}{2}\,y_{am,\,i} + O_2 \\ &= \frac{1}{2\,r}\,(\alpha^l_m)_0\,\mu_i + O_2 \end{aligned}\right\} \tag{4.36}$$

we get for the complex $\mathfrak{t}_i^{\,k}$ in (2.31)

$$\begin{aligned} \mathfrak{t}_i^{\,k} &= \frac{1}{\varkappa}\,\gamma^{km}{}_l\,\Delta^l{}_{mi} + O_3 \\ &= \frac{1}{4\,\varkappa\,r^2}\left[(\alpha^m_l)_0\,\mu^k - (\alpha^k_l)_0\,\mu^m\right](\alpha^l_m)_0\,\mu_i + O_3. \end{aligned}$$

Hence, outside the matter where $\mathfrak{T}_i^{\,k} = 0$,

$$\left.\begin{aligned} \mathsf{T}_i^{\,k} = \mathfrak{t}_i^{\,k} &= \frac{1}{4\,\varkappa\,r^2}\,(\alpha_{lm})_0\,(\alpha^{lm})_0\,\mu_i\,\mu^k + O_3 \\ &= \frac{2\,c_0{}^2}{\varkappa\,r^2}\,\mu_i\,\mu^k + O_3. \end{aligned}\right\} \tag{4.37}$$

Here, we have again used (4.34) and the relation

$$(\alpha_{ik})_0\,(\alpha^{ik})_0 = 8\,c_0{}^2 \tag{4.38}$$

following from (4.20) and (4.9).

For sufficiently large values of r where we can neglect the term O_3, the 3-vector density

$$\mathfrak{S}^{\varkappa} = -\,\mathfrak{t}_4^{\,\varkappa} = \frac{2\,c_0{}^2}{\varkappa\,r^2}\,n^{\varkappa}, \tag{4.39}$$

which represents the energy current, lies in the direction of increasing r. Now, let dx^λ and δx^μ denote two infinitesimal 3-vectors which are tangents to the sphere of radius r in the directions of increasing θ and φ, respectively. Then,

$$dx^\lambda = \{r\cos\theta\cos\varphi\, d\theta,\ r\cos\theta\sin\varphi\, d\theta,\ -r\sin\theta\, d\theta\} = r m^\lambda\, d\theta$$
$$\delta x^\mu = \{-r\sin\theta\sin\varphi\, d\varphi,\ r\sin\theta\cos\varphi\, d\varphi,\quad 0 \quad\} = r\sin\theta\, l^\mu\, d\varphi$$

and the quantity $dS_\varkappa = \delta_{\varkappa\lambda\mu}\, dx^\lambda\, \delta x^\mu$, representing the surface element spanned by these vectors, is

$$dS_\varkappa = n_\varkappa\, r^2 \sin\theta\, d\theta\, d\varphi. \tag{4.40}$$

Therefore, the energy flux through this surface element is, for sufficiently large values of r,

$$S d\theta\, d\varphi = \mathfrak{S}^\varkappa\, dS_\varkappa = \frac{2\, c_0{}^2}{\varkappa} \sin\theta\, d\theta\, d\varphi. \tag{4.41}$$

The total energy which per unit time leaves the sphere of radius r is obtained by integrating over all directions, i.e.,

$$-\frac{dH}{dt} = \iint S d\theta\, d\varphi = \frac{1}{2}\int_0^\pi c_0{}^2 \sin\theta\, d\theta. \tag{4.42}$$

This expression is in accordance with Bondi's equation (3.12), but the calculation has here been performed in the system S instead of in S', and we have to show that the same result holds in the system of coordinates used by Bondi et al. By means of the transformation law (2.34) for the energy-momentum complex it is now easy to calculate the quantity $\mathfrak{t}_4'^{\,k}$ in the system S'. Since the coordinate transformation (4.2) is of the type (2.36), $\mathfrak{t}_4{}^k$ transforms as a vector density.

Hence, by (4.5) and (4.37),

$$\mathfrak{t}_4'^{\,k} = r^2 \sin\theta\, \frac{\partial x'^{\,k}}{\partial x^m}\, \mathfrak{t}_4{}^m = \frac{2\, c_0{}^2}{\varkappa}\, \mu'^{\,k} \sin\theta + O_1 \tag{4.43}$$

with

$$\mu'^{\,k} = \frac{\partial x'^{\,k}}{\partial x^m}\, \mu^m = \{-1, 0, 0, 0\}. \tag{4.44}$$

Here we have used (4.14) and (4.9). Thus, we get for the 3-vector density

$$\mathfrak{S}'^{\,\varkappa} = -\mathfrak{t}_4'^{\,\varkappa} = \frac{2\, c_0{}^2 \sin\theta}{\varkappa}\, \delta_1^\varkappa + O_1 \tag{4.45}$$

and, for sufficiently large values of r, the energy flux through the surface element defined by (3.17) becomes

$$S' \, d\theta \, d\varphi = \mathfrak{S}'^{\varkappa} \, dS'_{\varkappa} = \frac{2 \, c_0{}^2}{\varkappa} \sin \theta \, d\theta \, d\varphi \, . \tag{4.46}$$

As one should expect, a comparison with (4.41) shows that the energy flux is the same in the system S' as in S, since the transformation (4.2) does neither change the system of reference nor the time scale.

On the other hand, Einstein's expression (2.10) does not have this important property. In the system S we get, by means of (4.30) and (4.25), for the Lagrangian $\mathfrak{L}_E$ defined by (2.8)

$$\left. \begin{aligned} \mathfrak{L}_E = & - \frac{1}{4 \, r^2} [(\alpha^{lk})_0 \, \mu_m + (\alpha^l_m)_0 \, \mu^k - (\alpha^k_m)_0 \, \mu^l] \\ & \times [(\alpha^m_k)_0 \, \mu_l + (\alpha^m_l)_0 \, \mu_k - (\alpha_{kl})_0 \, \mu^m] + O_3 = O_3 \end{aligned} \right\} \tag{4.47}$$

on account of (4.34) and (4.9). Further, since

$$\Gamma^s_{ms} = \frac{(\sqrt{-g})_{,m}}{\sqrt{-g}} = O_2$$

and

$$(\sqrt{-g} \, g^{lm})_{,i} = - y^{lm}{}_{,i} + O_2 = - \frac{(\alpha^{lm})_0}{r} \mu_i + O_2 \, ,$$

we get from (2.10)

$$\begin{aligned} \vartheta_i{}^k & = \frac{1}{2 \, \varkappa} \Gamma^k_{lm} (\sqrt{-g} \, g^{lm})_{,i} + O_3 \\ & = - \frac{1}{4 \, \varkappa r^2} [(\alpha^k_l)_0 \, \mu_m + (\alpha^k_m)_0 \, \mu_l - (\alpha_{lm})_0 \, \mu^k] (\alpha^{lm})_0 \, \mu_i + O_3 \\ & = \frac{1}{4 \, \varkappa r^2} (\alpha_{lm})_0 (\alpha^{lm})_0 \, \mu_i \, \mu^k + O_3 \end{aligned}$$

or, by (4.38),

$$\vartheta_i{}^k = \frac{2 \, c_0{}^2}{\varkappa \, r^2} \mu_i \, \mu^k + O_3 \, . \tag{4.48}$$

A comparison of (4.48) with (4.37) shows that Einstein's expression for the energy-momentum complex gives the correct value (4.41) for the energy flux when calculated in the system S. However, since $\vartheta_4{}^k$ transforms in an unphysical way under the transformation (4.2), it leads to the wrong result (3.20) for the radiated energy in the system S' adopted by Bondi et al.

The calculations of the present section have corroborated the conjecture of BONDI regarding the total energy flux from an axi-symmetric system expressed by the equation (3.12). In addition to that, our equation (4.41) gives the angular distribution of the energy flux which must be regarded as a measurable quantity, provided we can construct a receiver of gravitational energy which can be placed at large distances from the emitter in different directions. According to (4.41), the energy flux per unit solid angle is given by the square of the news function $c_0(t-r, \theta)$ and, by (3.14), we see that the energy flux must be zero in the direction of the symmetry axis.

For a detailed account of the angular distribution of the radiation from a given physical system we have to know the angular dependence of c_0. This requires a continuation of the solutions (3.2), (3.5) of the field equations at large distances into the interior of the system. So far this has been achieved exactly for a static system only. However, for a quasi-static system and sufficiently weak radiation, BONDI[1] found the following approximate expression for the news function $c(u, \theta)$:

$$c = \frac{1}{2} Q_{00} \sin^2 \theta, \tag{4.49}$$

where $Q(u)$ is the quadrupole moment of the system. According to this expression we should expect a steep maximum for the energy radiation in directions perpendicular to the axis of symmetry.

5. The Total Energy and Momentum

We begin this section by performing a calculation of the total energy which, as we shall see, leads to a verification of Bondi's conjecture that the quantity $m(u)$ in (3.11) represents the total mass or energy of the system. Also here it is most convenient to work in the system S instead of in S', but first we must show that the total energy is the same in both systems. According to (2.41), the energy contained in a large sphere of radius r at the time t is in the system S

$$H(u, r) = -\iint \mathfrak{U}_4^{4\lambda} \, dS_\lambda. \tag{5.1}$$

Similarly in S'

$$H'(u, r) = -\iint \mathfrak{U}_4'^{4\lambda} \, dS'_\lambda. \tag{5.2}$$

In this integral (5.1), r and t are constant, while in (5.2) r and u have to be kept constant.

Since $u = t - r$, the 2-surfaces over which the integrations in (5.1) and (5.2) are to be performed are identical. On account of (3.17), the integrant in the last integral is

$$\mathfrak{U}'^{4\lambda}_{4}\, dS'_{\lambda} = \mathfrak{U}'^{41}_{4}\, d\theta\, d\varphi. \tag{5.3}$$

Further, since $\mathfrak{U}_i^{kl}$ is a tensor density, we get, by means of (4.5), (4.13) and (4.14),

$$\left.\begin{aligned} \mathfrak{U}'^{41}_{4} &= J \frac{\partial x^r}{\partial x'^4} \frac{\partial x'^4}{\partial x^s} \frac{\partial x'^1}{\partial x^t} \mathfrak{U}_r^{st} \\ &= r^2 \sin\theta\, \delta^r_4\, \mu_s\, n_t\, \mathfrak{U}_r^{st}. \end{aligned}\right\} \tag{5.4}$$

Now, $\mathfrak{U}_r^{st}$ is antisymmetric in s and t and, according to (4.7), $\mu_s = \delta_s^{\,4} - n_s$. Therefore, by (5.3), (5.4) and (4.40),

$$\mathfrak{U}'^{4\lambda}_{4}\, dS'_{\lambda} = \mathfrak{U}_4^{4\tau}\, n_\tau\, r^2 \sin\theta\, d\theta\, d\varphi = \mathfrak{U}_4^{4\tau}\, dS_\tau, \tag{5.5}$$

which shows that the integrals in (5.1) and (5.2) are equal:

$$H'(u, r) = H(u, r). \tag{5.6}$$

The integrant in (5.1) can, by (4.40) and (4.7), be written

$$-\mathfrak{U}_4^{4\lambda}\, dS_\lambda = \mathfrak{U}_4^{4l}\, \mu_l\, r^2 \sin\theta\, d\theta\, d\varphi, \tag{5.7}$$

which shows that we have to calculate $\mathfrak{U}_4^{4l}\mu_l$ up to terms of the second power in $1/r$. More generally we have, by (2.30) and (4.6),

$$\mathfrak{U}_i^{kl}\, \mu_l = \frac{e^{2\beta}}{\varkappa} \left[\gamma^{kl}{}_i - \delta_i^{\,k}\, \Phi^l\, \mu_l + \mu_i\, \Phi^k\right]. \tag{5.8}$$

The calculation of $\gamma^{kl}{}_i$ and Φ^l for the tetrad field (4.29) is carried out in the Appendix (A.9–A.15) and gives the result

$$\mathfrak{U}_i^{kl}\, \mu_l = \frac{1}{2\varkappa r^2} \left\{ -4M + A)\, \mu_i\, \mu^k - (c_2 + 2c\cot\theta)\, m_i\, \mu^k \right\} + O_3. \tag{5.9}$$

By introduction of this expression with $i = k = 4$ into (5.7) and (5.1) we get

$$\left.\begin{aligned} H(u,r) &= \iint \mathfrak{U}_4^{\,4l}\,\mu_l\, r^2 \sin\theta\, d\theta\, d\varphi \\ &= \frac{1}{2\varkappa}\iint (4M - A)\sin\theta\, d\theta\, d\varphi + O_1 \end{aligned}\right\} \tag{5.10}$$

or, by means of (3.13) and (3.11) and putting $\varkappa = 8\pi$,

$$H(u,r) = m(u) + O_1. \tag{5.11}$$

For sufficiently large values of r we can neglect the last term in (5.11) and the remaining term, which is a function of u only, we shall *define* as the total energy H of a non-closed system. Thus, the total energy

$$H = m(u) = \frac{1}{2}\int_0^{\pi} M(u,\theta)\sin\theta\, d\theta \tag{5.12}$$

is just given by Bondi's expression (3.11).

We shall now perform a similar calculation of the total momentum of the system. As explained in section 2, this quantity is given by the simple expression (2.44) if, and only if, the system of coordinates is asymptotically rectilinear. This is obviously the case with S but not with S'. Thus, in S we get for the linear momentum contained in a large sphere of radius r, by (2.44), (4.40) and (5.9),

$$\left.\begin{aligned} P_\iota(u,r) &= \iint \mathfrak{U}_\iota^{\,4\lambda}\, dS_\lambda = -\iint \mathfrak{U}_\iota^{\,4l}\,\mu_l\, r^2\sin\theta\, d\theta\, d\varphi \\ &= \frac{1}{2\varkappa}\iint (4M - A)\, n_\iota \sin\theta\, d\theta\, d\varphi \\ &\quad - \frac{1}{2\varkappa}\iint (c_2 + 2c\cot\theta)\, m_\iota \sin\theta\, d\theta\, d\varphi + O_1. \end{aligned}\right\} \tag{5.13}$$

Thus,

$$P_\iota(u,r) = P_\iota(u) + O_1 \tag{5.14}$$

with

$$\left.\begin{aligned} P_\iota &= \frac{1}{2\varkappa}\iint (4M - A)\, n_\iota \sin\theta\, d\theta\, d\varphi \\ &\quad - \frac{1}{2\varkappa}\iint (c_2 + 2c\cot\theta)\, m_\iota \sin\theta\, d\theta\, d\varphi. \end{aligned}\right\} \tag{5.15}$$

Again we can neglect the last term O_1 in (5.14) for sufficiently large values of r and *define* the 'total momentum P_ι' of a non-closed system by the remaining term (5.15) which is a function of u only. Since M, A and c are independent of φ for an axi-symmetric system, while n_1, n_2, m_1, m_2 are proportional to $\cos\varphi$ or $\sin\varphi$, the integration over φ in (5.15) gives at once the value zero for the components of the total momentum in a direction perpendicular to the symmetry axis, as one should expect:

$$P_1 = P_2 = 0. \tag{5.16}$$

For the component along the symmetry axis we get, by (5.15), (3.8) and (4.7),

$$\begin{aligned} P_3 &= \frac{4\pi}{\varkappa}\int_0^\pi M\cos\theta\sin\theta\, d\theta \\ &-\frac{\pi}{\varkappa}\int_0^\pi \left[(c_2\sin\theta + 2\,c\cos\theta)_2\cos\theta - (c_2\sin\theta + 2\,c\cos\theta)\sin\theta\right] d\theta. \end{aligned} \tag{5.17}$$

The last integral is easily seen to be zero, for it is obviously equal to

$$\int_0^\pi \left[(c_2\sin\theta + 2\,c\cos\theta)\cos\theta\right]_2 d\theta = (c_2\sin\theta + 2\,c\cos\theta)\cos\theta\,\Big|_0^\pi = 0$$

on account of (3.14). Hence,

$$P_3 = \frac{1}{2}\int_0^\pi M(u,\theta)\cos\theta\sin\theta\, d\theta. \tag{5.18}$$

The equations (5.12), (5.16) and (5.18) show that the components of the 'total four-momentum'

$$P_i = \{P_\iota, -H\} \tag{5.19}$$

are

$$P_i = \left\{0,\, 0,\, \frac{1}{2}\int_0^\pi M\cos\theta\sin\theta\, d\theta,\, -\frac{1}{2}\int_0^\pi M\sin\theta\, d\theta\right\}. \tag{5.20}$$

For the time derivatives of these quantities we get, by means of (3.7) and (3.13),

$$\frac{dP_1}{dt} = \frac{dP_2}{dt} = 0 \tag{5.21}$$

$$\left.\begin{aligned} \frac{dP_4}{dt} = -\frac{dH}{dt} &= -\frac{1}{2}\int_0^\pi M_0 \sin\theta\, d\theta \\ &= \frac{1}{4\pi}\iint c_0^{\,2}\, d\Omega = \frac{1}{4\pi}\iint (\vec{\mathfrak{S}}\vec{n})\, r^2\, d\Omega, \end{aligned}\right\} \tag{5.22}$$

where $\vec{\mathfrak{S}}$ is the energy current density (4.39) and

$$d\Omega = \sin\theta\, d\theta\, d\varphi .$$

(5.22) shows that the energy is conserved.

Further,

$$\frac{dP_3}{dt} = \frac{1}{2}\int_0^\pi M_0 \cos\theta \sin d\theta = -\frac{1}{4\pi}\iint c_0^{\,2} \cos\theta\, d\Omega = -\iint \mathfrak{S}^3\, r^2\, d\Omega, \tag{5.23}$$

where $\mathfrak{S}^3$ is the component of the vector density (4.39) in the direction of the symmetry axis. Here we have used that the integral

$$\begin{aligned} \int_0^\pi A_0 \cos\theta \sin\theta\, d\theta &= \int_0^\pi [c_{22} \sin\theta\cos\theta + 3\, c_2 \cos^2\theta - 2\, c \sin\theta\cos\theta]_0\, d\theta \\ &= \int_0^\pi [c_{20} \sin\theta\cos\theta + c_0\,(1+\cos^2\theta)]_2\, d\theta \\ &= [c_{20}\sin\theta\cos\theta + c_0\,(1+\cos^2\theta)]\,\Big|_0^\pi = 0 \end{aligned}$$

on account of (3.14). The equations (5.21), (5.23) show that *the gravitational energy radiated administers a recoil to the system of the same amount as in the case of electromagnetic radiation.*

The relations (5.21)–(5.23) could also be obtained directly from (2.3). If we integrate this equation over the interior of a large sphere of radius r, we get

$$\frac{d}{dt}\iiint \mathsf{T}_i^{\,4}\, dx^1\, dx^2\, dx^3 = -\iiint \mathsf{T}_{i\ ,\varkappa}^{\,\varkappa}\, dx^1\, dx^2\, dx^3 = -\iint \mathsf{T}_i^{\,\varkappa}\, dS_\varkappa . \tag{5.24}$$

In virtue of (4.37) and (4.40) this gives, for sufficiently large values of r,

$$\left.\begin{aligned}\frac{dP_i}{dt} &= -\iint \frac{2\,c_0^2}{\varkappa r^2}\,\mu_i\,\mu^\varkappa\,n_\varkappa\,r^2\sin\theta\,d\theta\,d\varphi = \frac{1}{4\pi}\iint c_0^2\,\mu_i\,d\Omega \\ &= \left\{0\,,\,0\,,\,-\frac{1}{2}\int_0^\pi c_0^2\cos\theta\sin\theta\,d\theta\,,\,\frac{1}{2}\int_0^\pi c_0^2\sin\theta\,d\theta\right\}\end{aligned}\right\} \tag{5.25}$$

in accordance with (5.20)–(5.23).

In this section we have *defined* the 'total momentum and energy' as the quantities obtained from (5.11) and (5.14) by neglecting terms of the type O_1. This amounts to neglecting all terms of the type O_3 in $\mathfrak{U}_i^{4l}\mu_l$ in the surface integrals (5.10) and (5.13). Therefore, P_i will be equal to the momentum and energy contained in a sphere of radius r only if the terms O_3 occurring in the series expansion of $\mathfrak{U}_i^{4l}\mu_l$ are really negligible. For this to be true it is necessary that r is so large that the different terms of ascending powers in our series expansion of the metric correspond to decreasing orders of magnitude, i.e. we must have

$$1 \gg |\,y_{ik}\,| \gg |\,z_{ik}\,| \gg \cdots\cdots. \tag{5.26}$$

Further, we must require that r is so large that the last terms in (4.25), (4.26) are small compared with the first terms, i.e.

$$|\,(y_{ik})_0\,| \gg |\,y_{ik}\,| \cdot r^{-1} \tag{5.27}$$

and a similar relation for z_{ik}.

In view of the expressions (4.18)–(4.20) for y_{ik} and z_{ik} the condition (5.26) demands

$$\frac{|\,c\,|}{r} \ll 1,\quad \frac{|\,M\,|}{r} \ll 1,\quad \frac{|\,N\,|}{r} \ll |\,M\,|,\cdots\cdot. \tag{5.28}$$

Further, if λ is the order of magnitude of the wavelengths of the radiation emitted, we have

$$c_0 \sim \frac{c}{\lambda},\quad M_0 \sim \frac{M}{\lambda},\quad N_0 \sim \frac{N}{\lambda}\cdots\cdots\cdot \tag{5.29}$$

and (5.27) then means

$$r \gg \lambda. \tag{5.30}$$

Thus, the radius r has to be large compared with the wavelengths of the radiated waves, i.e. only if the surface of the sphere is lying in the 'wave zone' of the radiation will the momentum and energy contained in the sphere be equal to the 'total momentum and energy' P_i of the system. In all practical cases, c_0 and M_0 are very small quantities and it is easily seen that the conditions (5.28)–(5.30) are compatible with the relations (3.7), (3.8).

6. Invariance of P_i and of the Asymptotic Form of T_i^k under Tetrad Rotations

In section 2 it was shown that the complex T_i^k transforms according to Eq. (2.58) under tetrad rotations (2.51) and it is unchanged only if the rotation coefficients are constants. However, as also mentioned in section 2 and as we shall show now, the total four-momentum P_i as well as the asymptotic form of T_i^k are invariant under any rotation (2.51) for which also the new tetrads $\check{h}_i^{(a)}$ satisfy the boundary conditions A and B on p. 18. In our proof of this statement we shall again work in the system of coordinates S, where the boundary conditions have a particularly simple form, but since it is a statement regarding covariant quantities the proof is of course valid in any system of coordinates.

From (2.51) and (2.16) it follows that

$$\Omega^{(a)}{}_{(b)}(x) = \check{h}_i^{(a)} h^i_{(b)}. \tag{6.1}$$

Therefore, since the tetrads (4.29) have the limit

$$h^i_{(b)} \to \delta^i_b \quad \text{for} \quad r \to \infty, \tag{6.2}$$

the boundary condition A for $\check{h}_i^{(a)}$ yields

$$\Omega^{(a)}{}_{(b)}(x) \to \overset{(o)}{\Omega}{}^{(a)}{}_{(i)}\, \delta^i_b = \overset{(o)}{\Omega}{}^{(a)}{}_{(b)}, \tag{6.3}$$

where the coefficients $\overset{(o)}{\Omega}{}^{(a)}{}_{(b)}$ are constants.

However, T_i^k is unchanged under constant tetrad rotations so that we may choose $\overset{(o)}{\Omega}{}^{(a)}{}_{(b)} = \delta^a_b$ without any loss in generality. This means that $\overset{(o)}{\Omega}{}^{(a)}{}_{(b)}(x)$ must be of the form

$$\Omega^{(a)}{}_{(b)}(x) = \delta^a_b + \omega^a{}_b(x), \tag{6.4}$$

where

$$\omega^a{}_b(x) \to 0 \quad \text{for} \quad r \to \infty. \tag{6.5}$$

The indices a and b in $\omega^a{}_b$ are, of course, raised and lowered according to the same rule as in $\Omega^{(a)}{}_{(b)}$.

We shall now apply the boundary condition B to the new tetrad vector $\check{h}_i^{(a)}$, which requires that $\check{h}_i^{(a)} - \delta_i^a$ must have the same asymptotic behaviour as the metric quantities $\psi = g_{ik} - \eta_{ik}$. In the system S the boundary condition for ψ has the form of a Sommerfeld radiation condition, i. e. $\psi(u, r, \theta, \varphi)$ as a function of $u = t - r$, r, θ, and φ satisfies the condition

$$C. \qquad (r\psi)_1 = O_1' \to 0$$

for $r \to \infty$ under constant u, θ, φ.
Moreover, ψ and its first-order derivatives go to zero at least as $\frac{1}{r}$ for $r \to \infty$.

Now, condition B requires that C must hold also for $\psi = \check{h}_i^{(a)} - \delta_i^a$ which, on account of (6.1) and (6.4), implies that also the function $\psi = \omega_{ab}$ satisfies the condition C.

This means that $\omega_{ab}(u, r, \theta, \varphi)$ and its derivatives have the following asymptotic behaviour for $r \to \infty$: ω_{ab}, $(\omega_{ab})_0$, $(\omega_{ab})_2$ and $(\omega_{ab})_3$ go to zero at least as $1/r$, and $(\omega_{ab})_1$ goes to zero at least as $1/r^2$.

Symbolically this is expressed by

$$\left.\begin{aligned} \{\omega_{ab}, (\omega_{ab})_0, (\omega_{ab})_2, (\omega_{ab})_3\} &= O_1' \\ (\omega_{ab})_1 &= O_2', \end{aligned}\right\} \tag{6.6}$$

where O_n' means a term which goes to zero at least as $\frac{1}{r^n}$.

Otherwise the scalar functions ω_{ab} can be chosen completely arbitrarily apart, of course, from the orthogonality relations (2.52) which imply the following conditions:

$$\left.\begin{aligned} \omega_{ab} + \omega_{ab} + \omega^c{}_a\, \omega_{cb} &= 0 \\ \omega_a{}^c\, \omega_{bc} &= \omega^c{}_a\, \omega_{cb}. \end{aligned}\right\} \tag{6.7}$$

From the first of these equations and (6.6) we get for the symmetrical combination

$$\omega_{\{ab\}} \equiv \omega_{ab} + \omega_{ba} \tag{6.8}$$

$$\left.\begin{aligned} \{\omega_{\{ab\}}, (\omega_{\{ab\}})_0, (\omega_{\{ab\}})_2, (\omega_{\{ab\}})_3\} &= O_2' \\ (\omega_{\{ab\}})_1 &= O_3'. \end{aligned}\right\} \tag{6.9}$$

Further, from (4.24) with $\psi = \omega$

$$\omega_{ab,\,l} = (\omega_{ab})_0\,\mu_l + \left[(\omega_{ab})_1\,n_l + \frac{1}{r}\,(\omega_{ab})_2\,m_l + \frac{1}{r\sin\theta}\,(\omega_{ab})_3\,l_l\right], \tag{6.10}$$

where the bracket term is of the type O_2' on account of (6.6). Hence,

$$\omega_{\{ab\},\,l} = (\omega_{\{ab\}})_0\,\mu_l + O_3' = -(\omega^c{}_a\,\omega_{cb})_0\,\mu_l + O_3' \tag{6.11}$$

and, since

$$\mu_{l,\,i} = \frac{\partial \mu_l}{\partial x_i} = \frac{\partial^2 u}{\partial x^l\,\partial x^i} = O_1, \tag{6.12}$$

$$\omega_{\{ab\},\,l,\,i} = -(\omega^c{}_a\,\omega_{cb})_{00}\,\mu_l\,\mu_i + O_3'. \tag{6.13}$$

We shall now calculate the asymptotic form of the complex $\check{\mathsf{T}}_i{}^k$ corresponding to the tetrads $\check{h}_i^{(a)}$. By the asymptotic form we mean the expression for $\check{\mathsf{T}}_i{}^k$ obtained by neglecting all terms of type O_3, i.e. the expression which determines the energy radiated from the system.

According to (2.58) we have

$$\check{\mathsf{T}}_i{}^k - \mathsf{T}_i{}^k = Y_i{}^{kl}{}_{,\,l} \tag{6.14}$$

and we shall see that this quantity is really zero for large r if we neglect all terms O_3. To this power in $1/r$ we get, by (2.54), (4.29), (4.21) and (6.4)–(6.10),

$$\left.\begin{aligned} X^{kl}{}_i &= h^{(a)\,k}\,h^{(b)\,l}\,\Omega^{(c)}{}_{(a)}\,\Omega_{(cb),\,i} \\ &= \left(\eta^{ak} - \frac{1}{2}\,y^{ak}\right)\left(\eta^{bl} - \frac{1}{2}\,y^{bl}\right)(\delta^c_a + \omega^c{}_a)\,\omega_{cb,\,i} \\ &= \omega^{kl}{}_{,\,i} + \left[\omega^{rk}\,(\omega_r{}^l)_0 - \frac{1}{2}\,y^{rk}\,(\omega_r{}^l)_0 - \frac{1}{2}\,y^{rl}\,(\omega^k{}_r)_0\right]\mu_i + O_3'. \end{aligned}\right\} \tag{6.15}$$

Further, since $h = 1 + O_2$, we get from (2.56)

$$\left.\begin{aligned} \varkappa\,Y_i{}^{kl}{}_{,\,l} &= X^{kl}{}_{i,\,l} + X^k{}_{,\,i} - \delta^k_i\,X^l{}_{,\,l} + O_3' \\ &= Z_i{}^k - \frac{1}{2}\,\delta^k_i\,Z_l{}^l + O_3'. \end{aligned}\right\} \tag{6.16}$$

Here we have put

$$Z_i{}^k \equiv X^{kl}{}_{i,\,l} + X^k{}_{,\,i} = X^{kl}{}_{i,\,l} + X^{lk}{}_{l,\,i} \tag{6.17}$$

which gives

$$Z_i^i = X^{il}{}_{i,\,l} + X^{li}{}_{l,\,i} = 2\,X^l{}_{,\,l}. \tag{6.18}$$

Now, we get from (6.17), (6.15) and (6.6)–(6.12)

$$\left.\begin{aligned} Z_i^{\,k} &= \omega^{kl}{}_{,\,i,\,l} + \omega^{lk}{}_{,\,l,\,i} \\ &+ \left[\omega^{rk}(\omega_r{}^l)_0 - \frac{1}{2}y^{rk}(\omega_r{}^l)_0 - \frac{1}{2}y^{rl}(\omega^k{}_r)_0\right]_0 \mu_i\mu_l \\ &+ \left[\omega^{rl}(\omega_r{}^k)_0 - \frac{1}{2}y^{rl}(\omega_r{}^k)_0 - \frac{1}{2}y^{rk}(\omega^l{}_r)_0\right]_0 \mu_l\mu_i + O_3' \\ &= \omega^{\langle kl\rangle}{}_{,\,l,\,i} + \left[(\omega^{rk}\omega_r{}^l)_0 - \frac{1}{2}y_r{}^k(\omega^{\langle rl\rangle})_0\right]_0 \mu_i\mu_l + O_3'. \end{aligned}\right\} \tag{6.19}$$

Here we have used the relations

$$y^{rl}\mu_l = (y^{rl})_0\,\mu_l = 0 \tag{6.20}$$

following from the second equation (4.34).

By means of (6.9) and (6.13) the equations (6.19) and (6.16) give

$$Z_i^{\,k} = O_3', \quad Z_l^{\,l} = O_3' \tag{6.21}$$

and

$$Y_i^{\,kl}{}_{,\,l} = O_3'. \tag{6.22}$$

Thus, the asymptotic form of $\mathsf{T}_i^{\,k}$, obtained by neglecting all terms of the type O_3, is unchanged under all tetrad rotations which respect the boundary conditions A, B, and it is therefore uniquely given by the equation (4.37). In particular this holds for the gravitational energy emitted, which is uniquely given by the expressions (4.39)–(4.42).

We shall now show that also the total four-momentum P_i is invariant under these rotations. According to our definition of P_i and in view of (5.10), (5.13) and (2.55), the change in P_i under a tetrad rotation is given by

$$\left.\begin{aligned} \check{P} - P_i &= -\iint (\check{\mathfrak{U}}_i^{\,4\,l} - \mathfrak{U}_i^{\,4\,l})\,\mu_l r^2 \sin\theta\, d\theta\, d\varphi \\ &= -\iint Y_i^{\,4\,l}\mu_l r^2 \sin\theta\, d\theta\, d\varphi, \end{aligned}\right\} \tag{6.23}$$

where the integration is extended over the surface of a large sphere of radius r and *all terms of the type* O_1 *have to be discarded*. The calculation of $Y_i^{4\,l}\mu_l$, which runs along similar lines as the calculation of $Y_i^{\,kl}{}_{,\,l}$ in (6.15) –(6.22), is performed in the Appendix (A.16–A.22). The result is, neglecting terms of the type O_3,

$$\left.\begin{aligned}\varkappa Y_i^{4l}\mu_l &= \frac{(\omega^{4\lambda})_2}{r}(m_\lambda n_i - m_i n_\lambda) + \frac{(\omega^{4\lambda})_3}{r\sin\theta}(l_\lambda n_i - l_i n_\lambda)\\ &+ \delta_i^4\left[\frac{(\omega^{\varkappa\lambda})_2}{2r}(m_\varkappa n_\lambda - m_\lambda n_\varkappa) + \frac{(\omega^{\varkappa\lambda})_3}{2r\sin\theta}(l_\varkappa n_\lambda - l_\lambda n_\varkappa)\right].\end{aligned}\right\} \quad (6.24)$$

From the definitions (4.7) of n_i, m_i and l_i it follows that

$$\left.\begin{aligned} m_\varkappa n_\lambda - m_\lambda n_\varkappa &= -\delta_{\varkappa\lambda\nu} l^\nu \\ l_\varkappa n_\lambda - l_\lambda n_\varkappa &= \delta_{\varkappa\lambda\nu} m^\nu. \end{aligned}\right\} \quad (6.25)$$

Therefore, we get from (6.23)–(6.25) with $i = 4$

$$\check{P}_4 - P_4 = H - \check{H} = \frac{r\,\delta_{\varkappa\lambda\nu}}{2\varkappa}\int_0^\pi d\theta\int_0^{2\pi} d\varphi\,[(\omega^{\varkappa\lambda})_2\, l^\nu \sin\theta - (\omega^{\varkappa\lambda})_3\, m^\nu]. \quad (6.26)$$

By partial integrations of the first and second member with respect to θ and φ, respectively, we get, since the contributions from the boundaries cancel,

$$\check{P}_4 - P_4 = \frac{r\,\delta_{\varkappa\lambda\nu}}{2\varkappa}\int_0^\pi d\theta\int_0^{2\pi} d\varphi\,\omega^{\varkappa\lambda}\,[-l^\nu\cos\theta + (m^\nu)_3] = 0 \quad (6.27)$$

on account of (4.11). Similarly we get from (6.23)–(6.25) with $i = \iota$

$$\check{P}_\iota - \check{P}_\iota = 0\,. \quad (6.28)$$

Thus, also the total four-momentum P_i is invariant under all tetrad rotations which respect the boundary conditions A and B, and P_i is therefore uniquely given by the equation (5.20).

7. Transformation of P_i under Asymptotic Lorentz Transformations

We have seen that the total energy is invariant under the transformations (4.2) and (2.5) which leave the system of reference and the time scale unchanged. For more general coordinate transformations this will of course not be the case. We shall in particular investigate the transformation properties of P_i under transformations which, at large spatial distances from the system, have the form of a Lorentz transformation. For simplicity, we shall only consider such transformations of this type which lead from the system S with coordinates (4.1) to a system $\bar{S}$ with coordinates

$$\left.\begin{aligned}(\bar{x}^i) &= \{\bar{x}, \bar{y}, \bar{z}, \bar{t}\}\\ &= \{\bar{r}\sin\bar{\theta}\cos\bar{\varphi}, \bar{r}\sin\bar{\theta}\sin\bar{\varphi}, \bar{r}\cos\bar{\theta}, \bar{u}+\bar{r}\}\end{aligned}\right\} \tag{7.1}$$

for which the metric tensor $\bar{g}_{ik}$ asymptotically is of the same form as in (4.18)–(4.21). In part C and in the Appendix 3 of reference 2, A. W. K. Metzner has given the most general asymptotic form of a transformation of this kind. A special class of these transformations (a pure K-transformation) is given by

$$\left.\begin{aligned}r &= K\bar{r} + \bar{u}\left(\cosh\nu - \frac{1}{K}\right) + O_1\\ \theta &= \sin^{-1}\left(\frac{\sin\bar{\theta}}{K}\right) + \frac{\bar{u}}{\bar{r}}\frac{K'}{K^2} + O_2\\ \varphi &= \bar{\varphi}\\ u &= \frac{\bar{u}}{K} - \frac{\bar{u}^2}{2\,\bar{r}}\frac{(K')^2}{K^3} + O_2.\end{aligned}\right\} \tag{7.2}$$

Here,

$$K = K(\bar{\theta}) = \cosh\nu + \sinh\nu\cdot\cos\bar{\theta} > 0, \tag{7.3}$$

where ν is an arbitrary constant and

$$K' = \frac{dK}{d\bar{\theta}} = -\sinh\nu\sin\bar{\theta}. \tag{7.4}$$

From the second equation (7.2) it follows that

$$\sin\theta = \frac{\sin\bar{\theta}}{K}\left[1 - \frac{\bar{u}\,(K\cosh\nu - 1)}{\bar{r}K^2} + O_2\right] \tag{7.5}$$

$$\cos\theta = \frac{\sinh\nu + \cosh\nu\cos\bar{\theta}}{K} + O_1. \tag{7.6}$$

It is easily verified that the transformations (7.2), when written in terms of the variables (4.2), (7.1), asymptotically are of the form

$$\left.\begin{aligned} x &= \bar{x} + O_1, \qquad y = \bar{y} + O_1, \\ z &= \bar{z} \cosh \nu + \bar{t} \sinh \nu + O_1 \\ t &= \bar{t} \cosh \nu + \bar{z} \sinh \nu + O_1 \end{aligned}\right\} \tag{7.7}$$

which is a special Lorentz transformation with a relative velocity

$$v = \tan \nu \tag{7.8}$$

in the direction of the symmetry axis. Thus, far away from the matter system, the reference points of the system $\bar{S}$ are moving with the constant velocity v in the direction of the z-axis with respect to the system S.

The news function $\bar{c}(\bar{u}, \bar{\theta})$ and the mass aspect $\overline{M}(\bar{u}, \bar{\theta})$ in the system $\bar{S}$ are connected with the corresponding quantities in the system S by the following relations:

$$\left.\begin{aligned} &c = K\bar{c}, \; c_0 = K^2 \bar{c}_0, \; c_{00} = K^3 \bar{c}_{00}, \\ &M = K^3 \left[\overline{M} + f(\bar{u}, \bar{\theta})\right], \end{aligned}\right\} \tag{7.9}$$

where

$$f(\bar{u}, \bar{\theta}) = \frac{2\,\bar{u}\bar{c}_0}{K^2}\left[1 - K\cosh\nu + KK'\right] + \bar{u}\bar{c}_{02} K' + \frac{K'^2 \bar{u}^2 \bar{c}_{00}}{K^3}\left[K^2 - \frac{K}{2}\right] \tag{7.9a}$$

is a function of $\bar{u}$ and $\bar{\theta}$ which depends linearly on the derivatives $\bar{c}_0$, $\bar{c}_{00}$, $\bar{c}_{02}$ of the news function $\bar{c}(\bar{u}, \bar{\theta})$ with respect to $\bar{u}$ and $\bar{\theta}$. Therefore, for a system which does not radiate, i.e. for $\bar{c}_0 = 0$, the function $f(\bar{u}, \bar{\theta})$ vanishes. (Note that, if $\bar{c}_0 = 0$, then also $\bar{c}_0 = 0$ on account of (7.9)). Further, since we also in $\bar{S}$ have relations of the type (3.7), (3.8) we see that in this case also $\overline{M}_0 = 0$, i.e. $\overline{M} = \overline{M}(\bar{\theta})$ is a function of $\bar{\theta}$ only.

For the total momentum and energy in the system $\bar{S}$ we get, on the analogy of (5.20),

$$\overline{P}_i(\bar{u}) = \left\{0,\, 0,\, \frac{1}{2}\int_0^\pi \overline{M}(\bar{u}, \bar{\theta}) \cos\bar{\theta} \sin\bar{\theta}\, d\bar{\theta},\; -\frac{1}{2}\int_0^\pi \overline{M}(\bar{u}, \bar{\theta}) \sin\bar{\theta}\, d\bar{\theta}\right\}. \tag{7.10}$$

Thus,

$$P_1 = \overline{P}_1, \quad P_2 = \overline{P}_2 \tag{7.11}$$

i.e. the components in a spatial direction perpendicular to the relative motion are transformed as if P_i were a vector. However, the transformation of the components P_3 and P_4 is in general much more complicated. For instance, take the expression (5.20) for P_4,

$$P_4(u) = -\frac{1}{2}\int_0^\pi M(u,\theta)\sin\theta\, d\theta, \tag{7.12}$$

and introduce the variable $\bar{\theta}$ defined by the asymptotic form of (7.5), (7.6) as new variable of integration. Then we have

$$\sin\theta\, d\theta = \frac{\sin\bar{\theta}\, d\bar{\theta}}{K^2} \tag{7.13}$$

and, on account of (7.9) and (7.2),

$$P_4(u) = -\frac{1}{2}\int_0^\pi [\bar{M}(Ku,\bar{\theta}) + f(Ku,\bar{\theta})]\, K(\bar{\theta})\sin\bar{\theta}\, d\bar{\theta}, \tag{7.14}$$

where u during the integration has to be kept constant in the argument $\bar{u} = K \cdot u$ in the functions $M(\bar{u},\bar{\Theta})$, $f(\bar{u},\bar{\Theta})$.

Similarly, using also (7.6),

$$\left.\begin{aligned} P_3(u) &= \frac{1}{2}\int_0^\pi M(u,\theta)\cos\theta\sin\theta\, d\theta \\ &= \frac{1}{2}\int_0^\pi [\bar{M}(Ku,\bar{\theta}) + f(Ku,\bar{\theta})]\,[\sinh\nu + \cosh\nu\cos\bar{\theta}]\sin\bar{\theta}\, d\bar{\theta}. \end{aligned}\right\} \tag{7.15}$$

Since the variable $\bar{u} = K(\bar{\theta})u$ in $M(u,\bar{\theta})$ is varying over the range of integration in (7.14), (7.15), it is seen that there is in general no simple connection between (P_3, P_4) and $(\bar{P}_3, \bar{P}_4)$.

However, if the system for a certain period does not radiate, i.e. for $c_0 = \bar{c}_0 = 0$, then both P_i and $\bar{P}_i$ are constant in time. Further, we have then $f(\bar{u},\bar{\theta}) = 0$ and $\bar{M}_0 = 0$, i.e. $\bar{M} = \bar{M}(\bar{\theta})$ is independent of $\bar{u}$, and in this case we get from (7.14), (7.15) and (7.10)

$$\left.\begin{aligned} P_3 &= \frac{1}{2}\int_0^\pi \bar{M}(\bar{\theta})\,[\sinh\nu + \cosh\nu\cos\bar{\theta}]\sin\bar{\theta}\, d\bar{\theta} = \bar{P}_3\cosh\nu - \bar{P}_4\sinh\nu \\ P_4 &= -\frac{1}{2}\int_0^\pi \bar{M}(\bar{\theta})\,[\cosh\nu + \sinh\nu\cos\bar{\theta}]\sin\bar{\theta}\, d\bar{\theta} = \bar{P}_4\cosh\nu - \bar{P}_3\sinh\nu. \end{aligned}\right\} \tag{7.16}$$

Thus, for a non-radiative system, the quantities P_i transform as the covariant components of a free 4-vector under the asymptotic Lorentz transformations considered, i.e. under the K-transformations (7.2), (7.3).

This result is easily seen to hold for arbitrary asymptotic Lorentz transformations. To prove this statement we only need to show that $c_0 = 0$ implies

$$\mathsf{T}_i^{\,k} = O_4 \quad \text{for} \quad r \to \infty, \tag{7.17}$$

for, according to a well-known argument (see f. inst. reference 8), the 4-vector character of P_i is an immediate consequence of (2.3) and (7.17). Now, with $c_0 = 0$, (3.7) and (4.19) give

$$(\alpha_i^{\,k})_0 = 0 \tag{7.18}$$

and (4.24), (4.29)

$$h_{(a)\,k,\,l} = \frac{1}{2}(y_{ak})_0\,\mu_l + O_2 = O_2. \tag{7.19}$$

Further, by (4.32), (4.33), (4.36),

$$\left\{\Delta^i{}_{kl},\, \gamma_{ikl},\, \Phi_k,\, \Lambda_{ikl}\right\} = O_2 \tag{7.20}$$

and, since $\mathfrak{L}$ and $\mathsf{t}_i^{\,k}$ are homogeneous quadratic expressions in these quantities,

$$\left.\begin{aligned} \mathfrak{L} &= O_4 \\ \mathsf{T}_i^{\,k} = \mathsf{t}_i^{\,k} &= O_4, \end{aligned}\right\} \tag{7.21}$$

i.e. (7.17).

From (7.11) and (7.16) it follows that

$$\eta^{ik}\,P_i\,P_k = \eta^{ik}\,\overline{P}_i\,\overline{P}_k = -\,m_0^{\,2} \tag{7.22}$$

is an invariant, and we may assume that this quantity is negative so that we can define a real total rest mass m_0 of the system by (7.22). Then it is always possible by a suitable K-transformation to make $\overline{P}_\iota = 0$, and in this 'rest system' we have

$$\overline{H} = m_0. \tag{7.23}$$

For a radiative system we have seen that the total momentum and energy P_i does not transform in a simple way under the transformations (7.2), (7.7) and the same holds generally also for the gravitational energy and momentum radiated in a given time interval. However, if the radiation is going on a certain finite time only, so that $c_0(u, \theta)$ is different from zero in a time interval $u_1 < u < u_2$ but zero outside this interval, it is easily seen that the total gravitational four-momentum p_i emitted during the radiation period must again be a 4-vector under asymptotic Lorentz transformations. This follows at once from the law of conservation of energy and momentum which yields

$$p_i = P_i^{(1)} - P_i^{(2)}, \tag{7.24}$$

where $P_i^{(1)}$ and $P_i^{(2)}$ are the total four-momentum of the system before and after the radiation period, respectively. Since $P_i^{(1)}$ and $P_i^{(2)}$ are then 4-vectors the same holds of course for p_i.

An explicit expression for the gravitational four-momentum p_i is obtained by integrating (5.25) over the radiation period, i.e.

$$p_i = \left\{0,\, 0,\, \frac{1}{2}\int_{u_1}^{u_2} du \int_0^{\pi} c_0(u,\theta)^2 \cos\theta \sin\theta \, d\theta,\, -\frac{1}{2}\int_{u_1}^{u_2} du \int_0^{\pi} c_0(u,\theta)^2 \sin\theta \, d\theta\right\} \tag{7.25}$$

with an analogous expression for $\bar{p}_i$ in the system $\bar{S}$.

The 4-vector character of p_i is easily demonstrated directly by introducing the new variables of integration $\bar{u}$, $\overline{X\theta}$ obtained from (7.2), (7.4) by neglecting terms of the type O_1, i.e.

$$u = \frac{\bar{u}}{K}, \quad \sin\theta = \frac{\sin\bar{\theta}}{K}. \tag{7.26}$$

The corresponding Jacobian is, on account of (7.26) and (7.13),

$$\left.\begin{vmatrix} \dfrac{\partial u}{\partial \bar{u}} & \dfrac{\partial u}{\partial \bar{\theta}} \\[2ex] \dfrac{\partial \theta}{\partial \bar{u}} & \dfrac{\partial \theta}{\partial \bar{\theta}} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{K} & -\dfrac{\bar{u}K'}{K^2} \\[2ex] 0 & \dfrac{1}{K} \end{vmatrix} = \frac{1}{K^2}. \right\} \tag{7.27}$$

Thus, using (7.9) and (7.25),

$$\left.\begin{aligned} p_4 &= -\frac{1}{2}\iint c_0^2 \sin\theta \, du \, d\theta = -\frac{1}{2}\iint K\bar{c}_0^2 \sin\bar{\theta} \, d\bar{u} \, d\bar{\theta} \\ &= \bar{p}_4 \cosh\nu - \bar{p}_3 \sinh\nu. \end{aligned}\right\} \tag{7.28}$$

Similarly,

$$\left.\begin{aligned} p_3 &= \bar{p}_3 \cosh\nu - \bar{p}_4 \sinh\nu \\ p_1 &= \bar{p}_1 = 0, \quad p_2 = \bar{p} = 0. \end{aligned}\right\} \tag{7.29}$$

The emitted gravitational energy

$$h = -p_4 = \frac{1}{2}\iint c_0\,(u,\,\theta)^2 \sin\theta\, du\, d\theta \tag{7.30}$$

is always positive, and from (7.25) we see that

$$p_4^{\;2} > p_3^{\;2}. \tag{7.31}$$

Hence,

$$\eta^{ik}\, p_i\, p_k = p_3^{\;2} - p_4^{\;2} = -\mu^2 \tag{7.32}$$

with a real value for the 'rest mass' μ of the gravitational radiation. As far as energy and momentum are concerned, the loss in these quantities during the radiation period is exactly as if the system had emitted a particle of rest mass μ with the velocity

$$w = \left|\frac{p_3}{p_4}\right| < 1 \tag{7.33}$$

in the direction of the symmetry axis.

Although P_i in general is not exactly a 4-vector, this will be true with very good approximation in all practical cases. From the approximate expression (4.49) for c one can see that c_0 for all systems in nature is an extremely small quantity, so that we have

$$\bar{u}\,\bar{c}_0 \ll \overline{M} \tag{7.34}$$

for a large interval of $\bar{u}$. According to (5.25) and (7.9a) this means that P_i is only a slowly varying function of time and that the function f in (7.14), (7.15) can be neglected. Further we get by a Taylor expansion in $\bar{u} - u = u(K-1)$, and by means of the first equation (3.7) and (3.8) written in the system $\overline{S}$,

$$\left.\begin{aligned} \overline{M}\,(\bar{u},\,\bar{\theta}) &= \overline{M}\,(u,\,\bar{\theta}) + \overline{M}_0\,(u,\,\bar{\theta})\,(\bar{u} - u) \\ &= \overline{M}\,(u,\,\bar{\theta}) + O\,(\bar{u}\,\bar{c}_0) \end{aligned}\right\} \tag{7.35}$$

so that we can neglect the last term on account of (7.34). Then, the equations (7.14) and (7.15) are reduced to (7.16) with $\bar{u} = u$ in $\overline{P}_i\,(\bar{u})$ on the right-hand side of (7.16). This means that $P_i(u)$ transforms approximately as a 4-vector under asymptotic Lorentz transformations.

8. Approximate Plane Waves Emitted by a Distant Matter System

In this section we consider the gravitational radiation in a spatial region V of linear dimensions l at a large distance R from the matter system so that

$$l \ll R \tag{8.1}$$

and that V lies entirely in the wave zone defined by the relations (5.26)–(5.30). Then, it is easily seen that the solution (4.18)–(4.20) of Einstein's field equations inside V has the form of a weak field expansion

$$g_{ik} = \eta_{ik} + \overset{(1)}{y}_{ik} + \overset{(2)}{y}_{ik} + \cdot \cdot \cdot \cdot, \tag{8.2}$$

where the first approximation $\overset{(1)}{y}_{ik}$ represents a plane wave. Let us in particular consider the case where the region V is lying around the point $x = R$, $y = z = 0$ on the positive x-axis. Then it is convenient to introduce new coordinates

$$\left.\begin{aligned} (\bar{x}^i) &= \{\bar{x}, \bar{y}, \bar{z}, \bar{t}\}, \\ x = \bar{x} + R, \ y &= \bar{y}, \ z = \bar{z}, \ t = \bar{t} \end{aligned}\right\} \tag{8.3}$$

so that inside V

$$\left\{\frac{\bar{x}}{R}, \frac{\bar{y}}{R}, \frac{\bar{z}}{R}\right\} \ll 1 \tag{8.4}$$

$$r = \sqrt{(R+\bar{x})^2 + \bar{y}^2 + \bar{z}^2} = R + \bar{x} + O_1, \tag{8.5}$$

where O_n throughout this section means a term which is small of the n'th order. Further, if we put

$$\bar{u} = \bar{t} - \bar{x}, \tag{8.6}$$

we have

$$\left.\begin{aligned} u &= t - r = \bar{u} - R + O_1 \\ \cos\theta &= \frac{z}{r} = \frac{\bar{z}}{R} + O_2 = O_1, \quad \sin\theta = 1 + O_2 \\ \theta &= \frac{\pi}{2} + O_1 \\ \sin\varphi &= \frac{y}{r\sin\theta} = O_1, \quad \cos\varphi = 1 + O_2. \end{aligned}\right\} \tag{8.7}$$

Therefore, the quantities defined by (4.7) are of the form

$$\{n_i, m_i, l_i, \mu_i\} = \{\bar{n}_i, \bar{m}_i, \bar{l}_i, \bar{\mu}_i\} + O_1 \tag{8.8}$$

with

$$\left.\begin{aligned} \bar{n}_i &= \{\ 1,\ \ 0,\ \ 0,\ \ 0\} \\ \bar{m}_i &= \{\ 0,\ \ 0, -1,\ 0\} \\ \bar{l}_i &= \{\ 0,\ \ 1,\ \ 0,\ \ 0\} \\ \bar{\mu}_i &= \{-1, 0,\ \ 0,\ \ 1\} = \delta_i^4 - \bar{n}_i . \end{aligned}\right\} \tag{8.9}$$

The quantities (8.9) are constants which obviously satisfy the same relations (4.9), (4.12) as the quantities (4.7).

On account of (8.7) the functions $c(u,\theta)$, $M(u,\theta)$, $c_2(u,\theta)$ occurring in the expression (4.20) for α_{ik} have the following form inside V:

$$\left.\begin{aligned} &\{c(u,\theta), M(u,\theta), c_2(u,\theta)\} = \\ &\left\{c\left(\bar{u}-R, \frac{\pi}{2}\right), M\left(\bar{u}-R, \frac{\pi}{2}\right), c_2\left(\bar{u}-R, \frac{\pi}{2}\right)\right\} + O_1 . \end{aligned}\right\} \tag{8.10}$$

The quantities inside the curly brackets on the right-hand side of (8.10) are functions of $\bar{u}$, which we denote by $R\bar{c}(\bar{u})$, $R\overline{M}(\bar{u})$, $R\bar{c}_2(\bar{u})$, respectively. Then, inside V the quantity y_{ik} in (4.18) takes the form

$$y_{ik} = \overset{(1)}{y}_{ik} + O_2, \tag{8.11}$$

where

$$\overset{(1)}{y}_{ik} = 2\,\bar{c}(\bar{u})(\bar{m}_i\bar{m}_k - \bar{l}_i\bar{l}_k) + 2\,\overline{M}(\bar{u})\,\bar{\mu}_i\bar{\mu}_k + \bar{c}_2(\bar{u})(\bar{m}_i\bar{\mu}_k + \bar{\mu}_i\bar{m}_k) \tag{8.12}$$

is a function of $\bar{u} = t - \bar{x}$ only and therefore represents a plane wave travelling in the direction of the positive $\bar{x}$-axis. On the other hand, the term O_2 in (8.11) depends on $\bar{x}$, $\bar{y}$, $\bar{z}$ besides on $\bar{u}$ and, since this term is of the same order as z_{ik} in (4.18), we see that already the term of the second order $\overset{(2)}{y}_{ik}$ in (8.2) is *not* a pure plane wave.

Inside V the tetrad field (4.29) is of the form

$$h_{(a)i} = \eta_{ai} + \frac{1}{2}\overset{(1)}{y}_{ai} + O_2 . \tag{8.13}$$

Further, in the same region we get from (4.37) and (4.48)

$$\mathsf{T}_i^{\ k} = \mathsf{t}_i^{\ k} = \frac{2\,\bar{c}'(\bar{u})^2}{\varkappa}\bar{\mu}_i\bar{\mu}_k + O_3 \tag{8.14}$$

$$\theta_i^{\ k} = \vartheta_i^{\ k} = \frac{2\,\bar{c}'(\bar{u})^2}{\varkappa}\,\bar{\mu}_i\bar{\mu}^k + O_3, \tag{8.15}$$

where the prime means derivation with respect to $\bar{u}$, i.e.

$$\bar{c}'(\bar{u}) = \frac{d\bar{c}\,(\bar{u})}{d\bar{u}}. \tag{8.16}$$

Thus, in the system of coordinates $(\bar{x}_i)$, Einstein's expression $\Theta_i^{\ k}$ and the complex $\mathsf{T}_i^{\ k}$ give identical results if we neglect all terms of order O_3. The first order metric tensor

$$\bar{g}_{ik} = \eta_{ik} + \overset{(1)}{y}_{ik} \tag{8.17}$$

is a solution of the *linearized* field equations and it has the form

$$\bar{g}_{ik} = \begin{pmatrix} 1+2\,\overline{M} & 0 & \bar{c}_2 & -2\,\overline{M} \\ 0 & 1-2\,\bar{c} & 0 & 0 \\ \bar{c}_2 & 0 & 1+2\,\bar{c} & -\bar{c}_2 \\ -2\,\overline{M} & 0 & -\bar{c}_2 & -1+2\,\overline{M} \end{pmatrix}. \tag{8.18}$$

To the same order of magnitude we have

$$\bar{\mathfrak{g}}^{ik} = \eta^{ik} - \overset{(1)}{yik} \tag{8.19}$$

which satisfies the de Donder condition

$$\bar{\mathfrak{g}}^{ik}{}_{,\,k} = 0 \tag{8.20}$$

in accordance with the equation (4.28). This latter equation also shows that the de Donder condition is *not* satisfied in higher approximation. Introduction of the approximate metric tensor (8.17) into the expression (2.10) for $\vartheta_i^{\ k}$ gives of course just the formula (8.15). This is the usual procedure by which the energy flux in a weak plane gravitational wave has been calculated on the basis of Einstein's theory (13) and, on account of the accordance between $\Theta_i^{\ k}$ and $\mathsf{T}_i^{\ k}$ expressed in (8.14) and (8.15), this procedure seems to be justified.

However, it should be noted that the accordance between $\Theta_i^{\ k}$ and $\mathsf{T}_i^{\ k}$ holds in special coordinates only, and if we base our calculations on the complex $\Theta_i^{\ k}$ it is in general not easy to decide in which systems of coordinates this expression is valid. To illustrate this point let us again for a moment consider the case of a completely empty space, where we can use Lorentzian coordinates (X^i) with the metric tensor η_{ik}. In these coordinates the quantity $\vartheta_i^{\ k}$ in (2.10) is zero. By an infinitesimal coordinate transformation

$$X^i = x^i + \xi^i(x), \tag{8.21}$$

where the $\xi^i(x)$ are arbitrary functions of (x^i) which are small of the first order, we obtain for the metric tensor g_{ik} in the new system of coordinates an expression of the type (8.2) with the first order term

$$\overset{(1)}{g}_{ik} = \xi_{i,k} + \xi_{k,i} \tag{8.22}$$

$$\xi_i = \eta_{ik}\,\xi^k. \tag{8.23}$$

Then, a simple calculation shows that the quantity $\vartheta_i^{\ k}$ in (2.10) corresponding to the metric tensor g_{ik} in general is different from zero, i.e.

$$\vartheta_i^{\ k} \neq 0. \tag{8.24}$$

As shown in the Appendix (A.23)–(A.36), this arbitrariness in the value of Einstein's energy-momentum complex cannot be removed simply by requiring that we should use only harmonic systems of coordinates where the de Donder condition is satisfied. It is true that $\vartheta_i^{\ k}$ is equal to zero in all such systems (x^i) for which the quantities ξ^i in (8.21) are functions of $x^4 - x^1$ only, and this might indicate that Einstein's expression can be applied safely to those solutions of the field equations in empty space which are first order plane waves. On the other hand, the fact that a simple transformation of coordinates in a flat space may create a $\vartheta_i^{\ k}$ of the same order of magnitude as a 'real' gravitational field makes one feel uneasy in applying Einstein's expression in general.

The just mentioned difficulty does not arise with the complex $\mathsf{T}_i^{\ k}$, for in a completely empty space we have in all systems of coordinates exactly

$$\mathsf{T}_i^{\ k} = \mathsf{t}_i^{\ k} = 0 \tag{8.25}$$

on account of the condition (2.60) imposed in this case. For a weak gravitational field of the type (8.18) we have then also to require that $h_i^{(a)}$ deviates from the values (2.64) or (2.65), valid in a completely empty space, by a quantity which is small of the first order. This requirement is of course satisfied by the expression (8.13) which is symmetrical in a and i. If we introduce this tetrad field into the expression (2.31) for $\mathrm{t}_i^{\ k}$ we get of course just the formula (8.14). Now, the proof in section 6 for the invariance of $\mathsf{T}_i^{\ k}$ under tetrad rotations of the type (6.4)–(6.7) leads to the result that $\mathsf{T}_i^{\ k} = \mathrm{t}_i^{\ k}$ inside V is unchanged under all rotations of the tetrads (8.13) with coefficients $\Omega_{(ab)}$ of the form

$$\Omega_{(ab)} = \eta_{ab} + \overline{\omega}_{ab}(\bar{u}), \tag{8.26}$$

where the quantity $\overline{\omega}_{ab}(\bar{u})$ is any function of $\bar{u} = t - \bar{x}$ which is small of the first order and antisymmetric in the indices a and b. The invariance of the second order expression for $\mathrm{t}_i^{\ k}$ under such rotations is shown explicitly in the Appendix (A.37)–(A.45). A finite rotation with constant coefficients $\overset{(0)}{\Omega}_{(ab)}$ will of course also leave $\mathsf{T}_i^{\ k}$ unchanged.

An infinitesimal coordinate transformation $(\bar{x}^i) \to (x'^i)$ of the type

$$x'^i = \bar{x}^i + \xi^i(\bar{x}) \tag{8.27}$$

changes the complex $\mathsf{T}_i^{\ k}$ given by (8.14) into

$$\mathsf{T}'^{\ k}_i = \mathsf{T}_i^{\ k} - \xi^r_{\ ,i,n} \overset{(1)}{\mathfrak{U}}{}_r^{\ kn}. \tag{8.28}$$

Here, in using (2.34), we have neglected all terms of order O_3 and

$$\overset{(1)}{\mathfrak{U}}{}_r^{\ kn} = \frac{1}{2\varkappa}\left[\left(\overset{(1)}{y}{}_r^{\ n}\right)' \overline{\mu}^k - \left(\overset{(1)}{y}{}_r^{\ k}\right)' \overline{\mu}^n\right] \tag{8.29}$$

is the first order expression of the superpotential (2.30) corresponding to the tetrads (8.13). If $\xi^i = \xi^i(\bar{u})$ is a function of $\bar{u} = t - \bar{x}$ only, we have

$$\xi^r_{\ ,i,n} = (\xi^r)'' \,\overline{\mu}_i \overline{\mu}_n \tag{8.30}$$

and, since

$$\overline{\mu}_n \overset{(1)}{\mathfrak{U}}{}_r^{\ kn} = 0, \tag{8.31}$$

we have in this case

$$\mathsf{T}'^{\ k}_i = \mathsf{T}_i^{\ k}. \tag{8.32}$$

Thus, to the second order the energy-momentum complex is unchanged under infinitesimal coordinate transformations where ξ^i is a function of $\bar{u}$ only. By a transformation of this type the metric (8.18) can be brought into one of its two standard forms (13). If we choose ξ_i of the form

$$\xi_i = \chi(\bar{u})\,\overline{\mu}_i + \Phi(\bar{u})\,\bar{m}_i, \tag{8.33}$$

where χ and Φ are functions of $\bar{u}$ only, we have

$$\xi_{i,\,k} = \chi'(\bar{u})\,\bar{\mu}_i\,\bar{\mu}_k + \Phi'(\bar{u})\,\bar{m}_i\,\bar{\mu}_k. \tag{8.34}$$

Then, we get for the metric tensor in the new system

$$\left.\begin{aligned} g'_{ik} &= \eta_{ik} + \overset{(1)}{y'_{ik}}, \\ \overset{(1)}{y'_{ik}} &= \overset{(1)}{y_{ik}} - \xi_{i,\,k} - \xi_{k,\,i} = 2\,\bar{c}\,(\bar{m}_i\,\bar{m}_k - \bar{l}_i\,\bar{l}_k) \\ &+ 2\,(\overline{M} - \chi')\,\bar{\mu}_i\,\bar{\mu}_k + (\bar{c}_2 - \Phi')\,(\bar{m}_i\,\bar{\mu}_k + \bar{m}_k\,\bar{\mu}_i). \end{aligned}\right\} \tag{8.35}$$

With

$$\chi(\bar{u}) = \int \overline{M}(\bar{u})\,d\bar{u}, \quad \Phi(\bar{u}) = \int \bar{c}_2(\bar{u})\,d\bar{u} \tag{8.36}$$

this gives

$$\overset{(1)}{y'_{ik}} = 2\,\bar{c}\,(\bar{m}_i\,\bar{m}_k - \bar{l}_i\,\bar{l}_k) \tag{8.37}$$

or

$$\left. g'_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - 2\,\bar{c} & 0 & 0 \\ 0 & 0 & 1 + 2\,\bar{c} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \right\} \tag{8.38}$$

Thus, the new system of coordinates is a 'rectangular' system and since

$$u' \equiv t' - x' = \bar{t} - \chi - (\bar{x} - \chi) = \bar{t} - \bar{x} = \bar{u} \tag{8.39}$$

we have

$$\bar{c} = \bar{c}(\bar{u}) = \bar{c}(u'). \tag{8.40}$$

On account of (8.32), the energy-momentum complex $\mathsf{T}'^{\,k}_{i}$ in this system is also given by (8.14).

The components of the tetrads (8.13) in the system $(x'^{\,i})$ are

$$\left.\begin{aligned} h'_{(a)\,i} &= \frac{\partial x^k}{\partial x'^{\,i}}\,h_{(a)\,k} = (\delta^k_i - \xi^k{}_{,\,i})\left(\eta_{ak} + \frac{1}{2}\overset{(1)}{y_{ak}}\right) \\ &= \eta_{ai} + \bar{c}\,(\bar{m}_a\,\bar{m}_i - \bar{l}_a\,\bar{l}_i) + \frac{1}{2}\,\bar{c}_2\,(\bar{\mu}_a\,\bar{m}_i - \bar{m}_a\,\bar{\mu}), \end{aligned}\right\} \tag{8.41}$$

an expression which is not symmetrical in a and i. By a rotation of the tetrads of the form (8.26), which leaves $\mathsf{T}_i^{\,k}$ unchanged, we get

$$\check{h}'_{(a)\,i} = (\delta^b_a + \varpi_a{}^b(\bar{u}))\,h'_{(b)\,i} = h'_{(a)\,i} + \varpi_{ai}(\bar{u}). \tag{8.42}$$

Thus, if we choose

$$\bar{\omega}_{ab}(\bar{u}) = -\frac{1}{2}\bar{c}_2(\bar{u})(\bar{\mu}_a \bar{m}_b - \bar{m}_a \bar{\mu}_b) = -\bar{\omega}_{ba}, \tag{8.43}$$

$\check{h}'_{(a)\,i}$ takes the form

$$\check{h}'_{(a)\,i} = \eta_{ai} + \bar{c}(\bar{m}_a \bar{m}_i - \bar{l}_a \bar{l}_i) = \eta_{ai} + \frac{1}{2}\overset{(1)}{y}_{ai}, \tag{8.44}$$

i.e. also in the rectangular system the tetrads can be chosen symmetrical in the tetrad- and vector indices.

Finally, a spatial rotation about the x'^1-axis through the angle $\pi/4$ leads to a system of coordinates (x''^i) in which the metric has the other standard form (13):

$$g''_{ik} = \eta_{ik} + \overset{(1)}{y}{}''_{ik}, \tag{8.45}$$

where

$$\overset{(1)}{y}{}''_{23} = \overset{(1)}{y}{}''_{32} = 2\,\bar{c} \tag{8.46}$$

and all the other components are zero. If we perform the same constant rotation of the tetrads $\check{h}'_{(a)\,i}$, which leaves $\mathsf{T}_i^{\,k}$ unchanged, we see that also in this system the tetrads can be chosen symmetrical in a and i, i.e. for the rotated tetrads we have

$$\check{h}''_{(a)\,i} = \eta_{ai} + \frac{1}{2}\overset{(1)}{y}{}''_{ai}. \tag{8.47}$$

Let us now consider a sandwich wave, where $\bar{c}'(\bar{u}) = 0$ outside an interval $\bar{u}_1 < \bar{u} < \bar{u}_2$. The momentum and energy per unit area in the $(\bar{y}, \bar{z})$-plane of the system $(\bar{x}^i)$ is then, in virtue of (8.14) and (8.9),

$$\bar{p}_i = \int \mathrm{t}_i^{\,4}\, d\bar{x} = -\frac{2}{\varkappa}\bar{\mu}_i \int_{\bar{u}_1}^{\bar{u}_2} (\bar{c}')^2\, d\bar{u}, \tag{8.48}$$

i.e.

$$\bar{p}_i = \left\{ \frac{2}{\varkappa}\int_{\bar{u}_1}^{\bar{u}_2} \bar{c}'(\bar{u})^2\, d\bar{u},\ 0,\ 0,\ -\frac{2}{\varkappa}\int_{u_1}^{\bar{u}_2} \bar{c}'(\bar{u})^2\, d\bar{u} \right\}. \tag{8.49}$$

It is also clear that $\bar{p}_i$ transforms as a 4-vector under Lorentz transformations

$$x'^i = a^i_{\ k}\bar{x}^k \tag{8.50}$$

for, according to (2.34), T_i^k transforms as a tensor under such transformations. Let us briefly consider a special Lorentz transformation in the direction of the $\bar{x}$-axis, i.e.

$$\left.\begin{aligned} \bar{x} &= x' \cosh \nu + t' \sinh \nu \\ \bar{t} &= t' \cosh \nu + x' \sinh \nu \\ \bar{y} &= y', \ \bar{z} = z'. \end{aligned}\right\} \tag{8.51}$$

Then,

$$\left.\begin{aligned} \bar{u} = \bar{t} - \bar{x} &= (\cosh \nu - \sinh \nu)\, u' \\ u' &\equiv t' - x', \end{aligned}\right\} \tag{8.52}$$

and

$$\mathsf{T}'^{\,k}_{i} = \frac{2}{\varkappa}\, \bar{c}' (u')^2 \mu'_i \mu'^k \tag{8.53}$$

with

$$\mu'_i = \frac{\partial \bar{x}^k}{\partial x'^i} \bar{\mu}_k = -\frac{\partial \bar{x}^1}{\partial x'^i} + \frac{\partial \bar{x}^4}{\partial \bar{x}'^i} = (\cosh \nu - \sinh \nu)\, \bar{\mu}_i. \tag{8.54}$$

Thus,

$$\left.\begin{aligned} p'_i = \int \mathsf{T}'^4_i \, dx' = \int \mathsf{T}'^4_i \, du' &= -\frac{2}{\varkappa} (\cosh \nu - \sinh \nu)^2 \bar{\mu}_i \int \bar{c}\, (\bar{u})^2 \, du' \\ &= -(\cosh \nu - \sinh \nu) \frac{2}{\varkappa} \bar{\mu}_i \int_{\bar{u}_1}^{\bar{u}_2} \bar{c}'\, (\bar{u})^2 \, d\bar{u}, \end{aligned}\right\} \tag{8.55}$$

or

$$\left.\begin{aligned} p'_1 &= \bar{p}_1 \cosh \nu + \bar{p}_4 \sinh \nu, \quad p'_2 = p_2, \\ p'_3 &= \bar{p}_3, \quad p'_4 = \bar{p}_1 \sinh \nu + \bar{p}_4 \cosh \nu \end{aligned}\right\} \tag{8.56}$$

in accordance with the transformation law for a four-momentum vector.

We can now always combine the Lorentz transformations (8.50) of the coordinates with the corresponding rotation of the tetrads, i.e.

$$\check{h}_{(a)\,i} = \Omega_{(a)}{}^{(b)}\, h_{(b)\,i} = a_a{}^b\, h_{(b)\,i} \tag{8.57}$$

which leaves the T_i^k unchanged. Then it is clear that the components of the rotated tetrads in the transformed system of coordinates are again given by

$$\check{h}'_{(a)\,i} = \eta_{ai} + \frac{1}{2} \overset{(1)}{y'_{ik}}, \tag{8.58}$$

where $\overset{(1)}{y'_{ik}}$ is the first order term of the transformed metric tensor g'_{ik}.

As regards energy and momentum, the wave packet of gravitational radiation with the four-momentum $\bar{p}_i$ is quite analogous to a corresponding electromagnetic wave. From (8.49) we see that the invariant norm of the four-momentum is zero, i.e.

$$\bar{p}_i \bar{p} = p_i' p'^i = 0, \tag{8.59}$$

which corresponds to a vanishing "rest mass" of the packet.

Appendix

We start by establishing a few relations needed in the following calculations. They are all consequences of the expressions (4.20) for α_{ik} and β_{ik} and of the equations (4.9)–(4.11). First, we quote again the relations (4.34), (4.38), and some immediate consequences of these equations:

$$\left.\begin{aligned} &\alpha_i^i = 0, \quad \alpha^{ik}\mu_k = 0, \quad \alpha^{ik} n_k = \alpha^{i4} \\ &(\alpha^{ik})_0 \mu_k = 0, \quad (\alpha^{ik})_2 \mu_k = -\alpha^{ik}(\mu_k)_2 = \alpha^{ik} m_k, \\ &(\alpha^{ik})_3 \mu_k = -\alpha^{ik}(\mu_k)_3 = \sin\theta\, \alpha^{ik} l_k, \ (\alpha_{ik})_0 (\alpha^{ik})_0 = 8\, c_0{}^2. \end{aligned}\right\} \tag{A.1}$$

Similarly, we have

$$\left.\begin{aligned} &\alpha_{ik}\alpha^{ik} = 8\, c^2, \quad (\alpha_{ik})_0 \alpha^{ik} = 8\, cc_0 \\ &\beta_i^i = 3\, c^2, \quad (\beta_i^i)_0 = 6\, cc_0 \\ &\beta^{ik}\mu_k = -\frac{c^2}{2}\mu^i, \quad (\beta^{ik})_0 \mu_k = -cc_0\, \mu^i. \end{aligned}\right\} \tag{A.2}$$

Further,

$$\left.\begin{aligned} &\alpha^{ik} m_k = 2\, c\, m^i + (c_2 + 2\, c\cot\theta)\,\mu^i \\ &\alpha^{ik} l_k = -2\, cl^i. \end{aligned}\right\} \tag{A.3}$$

By differentation of these equations with respect to θ and φ, respectively, and by using (4.11) and (A.1), we obtain

$$\left.\begin{aligned} &(\alpha^{ik})_2 m_k = \alpha^{i4} + (c_2 - 2\, c\cot\theta)\, m^i + \left(c_{22} + 2\, c_2 \cot\theta - \frac{2\, c}{\sin^2\theta}\right)\mu^i - 2\, cn^i \\ &\frac{1}{\sin\theta}(\alpha^{ik})_3 l_k = \alpha^{i4} + \cot\theta\,[4\, cm^i + (c_2 + 2\, c\cot\theta)\,\mu^i] + 2\, cn_i. \end{aligned}\right\} \tag{A.4}$$

Hence,

$$(\alpha^{ik})_2 m_k + \frac{1}{\sin\theta}(\alpha^{ik})_3 l_k = 2\,\alpha^{i4} + (c_2 + 2\, c\cot\theta)\, m^i + A\,\mu^i$$

with A given by (3.8). Further, since

$$\alpha^{i\,4} = -2\,M\mu^i - (c_2 + 2\,c\cot\theta)\,m^i, \tag{A.5}$$

this may be written

$$(\alpha^{ik})_2\,m_k + \frac{1}{\sin\theta}(\alpha^{ik})_3\,l_k = \alpha^{i\,4} + (A - 2\,M)\,\mu^i. \tag{A.6}$$

Now, we get from (4.25), (A.1), and (A.6)

$$\left.\begin{aligned} y^{ik}{}_{,k} &= \left(\frac{\alpha^{ik}}{r}\right)_{,k} = \frac{(\alpha^{ik})_0}{r}\mu_k + \frac{-\alpha^{i\,4} + \alpha^{i\,4} + (A - 2\,M)\,\mu^i}{r^2} \\ &= \frac{(A - 2\,M)\,\mu^i}{r^2} \end{aligned}\right\} \tag{A.7}$$

and, therefore, by (4.27), (A.1), (A.2),

$$\left.\begin{aligned} \mathfrak{g}^{ik}{}_{,k} &= -(y^{ik})_{,k} - \frac{(\beta^{ik})_0 - (\alpha^{ir}\,\alpha^k_r)_0 + cc_0\,\eta^{ik}}{r^2}\mu_k + O_3 \\ &= \frac{2\,M - A}{r^2}\mu^i + O_3, \end{aligned}\right\} \tag{A.8}$$

i.e. the equation (4.28) in the text.

We shall now calculate the tensor $\gamma^{kl}{}_i$ and the vector Φ^l for the tetrad field (4.29) up to terms of the second power in $1/r$. From (4.31) and (4.29) we get

$$\begin{aligned} \gamma_{rsi} &= h^{(a)}_r\,h_{(a)\,s,\,i} - \Gamma_{r,\,si} \\ &= \left(\delta^a_r + \frac{1}{2}y^a_r\right)\frac{1}{2}\left[y_{as,\,i} + z_{as,\,i} - \frac{1}{4}(y_{at}\,y^t_s)_{,\,i}\right] - \frac{1}{2}(g_{rs,\,i} + g_{ri,\,s} - g_{si,\,r}) + O_3 \\ &= \frac{1}{2}\left[g_{is,\,r} - g_{ir,\,s} + \frac{1}{4}(y_{rt}\,y^t_{s,\,i} - y^t_{r,\,i}\,y_{st})\right] + O_3. \end{aligned}$$

On account of (4.25), (4.26), this gives

$$\left.\begin{aligned} \gamma_{rsi} &= A_{rsi} - A_{sri} + O_3 \\ A_{rsi} &= \frac{(\alpha_{is})_0}{2\,r}\mu_r + \frac{1}{2\,r^2}\left[-\alpha_{is}\,n_r + (\alpha_{is})_2\,m_r + (\alpha_{is})_3\,l_r/\sin\theta + (\beta_{is})_0\,\mu_r + \frac{1}{4}\alpha_{rt}\,(\alpha^t_s)_0\,\mu^i\right] \end{aligned}\right\} \tag{A.9}$$

Finally, by raising the two first indices by means of (4.21),

$$\gamma^{kl}{}_i = g^{kr} g^{ls} \gamma_{rsi}$$
$$= \eta^{kr} \eta^{ls} \gamma^{\prime si} - \frac{1}{2 r^2} (\eta^{kr} \alpha^{ls} + \eta^{ls} \alpha^{kr}) [(\alpha_{is})_0 \mu_r - (\alpha_{ir})_0 \mu_s] + O_3.$$

Thus,

$$\left.\begin{aligned} \gamma^{kl}{}_i = & \frac{1}{2 r} [(\alpha_i^l)_0 \mu^k - (\alpha_i^k)_0 \mu^l] \\ & + \frac{1}{2 r^2} \Big\{ - \alpha_i^l n^k + \alpha_i{}^k n^l + (\alpha_i{}^l)_2 m^k - (\alpha_i{}^k)_2 m^l \\ & + \frac{1}{\sin \theta} [(\alpha_i{}^l)_3 l^k - (\alpha_i{}^k)_3 l^l] + \mu^k [(\beta_i{}^l)_0 - (\alpha_{ir})_0 \alpha^{lr}] \\ & - \mu^l [(\beta_i^k)_0 - (\alpha_{ir})_0 \alpha^{kr}] + \frac{1}{4} [\alpha_r^k (\alpha^{rl})_0 - \alpha^l{}_r (\alpha^{rk})_0] \mu_i \Big\} + O_3. \end{aligned}\right\} \quad (A.10)$$

By contraction of this expression with respect to i and k, we get by means of (A.1), (A.2), and (A.6) for the vector Φ^l

$$\left.\begin{aligned} \Phi^l = \gamma^{il}{}_i = & \frac{1}{2 r^2} \Big\{ - \alpha^{l\,4} + (\alpha_i{}^l)_2 m^i + \frac{1}{\sin \theta} (\alpha^l{}_i)_3 l^i \\ & - c c_0 \mu^l - 6 c c_0 \mu^l + \mu^l (\alpha_{rs})_0 \alpha^{rs} \Big\} + O_3 \\ = & \frac{1}{2 r^2} [A - 2 M + c c_0] \mu^l + O_3. \end{aligned}\right\} \quad (A.11)$$

From this we see that

$$\Phi^l \mu_l = O_3 \tag{A.12}$$

and

$$\Phi^k \mu_i = \frac{1}{2 r^2} [A - 2 M + c c_0] \mu^k \mu_i + O_3. \tag{A.13}$$

Further, from (A.10), by applying (A.1), (A.2), and (4.9),

$$\gamma^{kl}{}_i \mu_l = \frac{1}{2 r^2} \{ - \alpha_i^k + \alpha_i^l m_l m^k + \alpha_i^l l_l l^k - \mu^k c c_0 \mu_i \} + O_3$$

or, using (A.3),

$$\gamma^{kl}{}_i \mu_l = - \frac{1}{2 r^2} \{ (2 M + c c_0) \mu_i \mu^k + (c_2 + 2 c \cot \theta) m_i \mu^k \} + O_3. \tag{A.14}$$

Therefore, since $h = e^{2\beta} = 1 + O^2$, we get from (5.8), (A.12)–(A.14),

$$\mathfrak{U}_i^{\ kl}\mu_l = \frac{1}{2\varkappa r^2}\{(-4M + A)\mu_i\mu^k - (c_2 + 2c\cot\theta)\, m_i\mu^k\} + O_3, \qquad \text{(A.15)}$$

which is the formula (5.9) in the text.

Our next task is to prove the formula (6.24) for the quantity $\varkappa Y_i^{\ 4l}\mu_l$. To the order required we have, just as in (6.16),

$$\varkappa Y_i^{\ kl}\mu_l = Z_i^{\ k} - \frac{1}{2} Z_l^{\ l}\delta_i^k + O_3', \qquad \text{(A.16)}$$

where now

$$Z_i^{\ k} = X^{kl}_{\ \ i}\mu_l + X^{lk}_{\ \ l}\mu_i. \qquad \text{(A.17)}$$

From (6.15) and (6.20) we get for this quantity

$$\left.\begin{aligned} Z_i^{\ k} = \omega^{kl}_{\ \ ,i}\mu_l + \omega^{lk}_{\ \ ,l}\mu_i + [\omega^{rk}(\omega_r^{\ l})_0 + \omega^{rl}(\omega_r^{\ k})_0]\,\mu_i\mu_l \\ -\frac{1}{2} y_r^k(\omega^{rl} + \omega^{lr})_0\mu_i\mu_l + O_3'. \end{aligned}\right\} \qquad \text{(A.18)}$$

Here, the last term but one is also of the order of O_3' on account of (6.9) and, by means of (6.10), we get

$$\left.\begin{aligned} Z_i^{\ k} &= (\omega^{kl} + \omega^{lk} + \omega^{rk}\omega_r^{\ l})_0\,\mu_i\mu_l \\ &+ (\omega^{kl})_1\, n_i\mu_l + (\omega^{lk})_1\, n_l\mu_i \\ &+ \frac{1}{r}[(\omega^{kl})_2\, m_i\mu_l + (\omega^{lk})_2\, m_l\mu_i] \\ &+ \frac{1}{r\sin\theta}[(\omega^{kl})_3\, l_i\mu_l + (\omega^{lk})_3\, l_l\mu_i] + O_3'. \end{aligned}\right\} \qquad \text{(A.19)}$$

The first term in this expression is zero on account of (6.7), and in the remaining terms ω^{kl} can be treated as antisymmetric in k and l in virtue of (6. 9). Finally, since $\mu_i = \delta_i^4 - n_i$, we get

$$\left.\begin{aligned} Z_i^{\ k} &= (\omega^{k4})_1\, n_i + \frac{1}{r}(\omega^{k4})_2\, m_i + \frac{1}{r\sin\theta}(\omega^{k4})_3\, l_i \\ &+ \delta_i^4\left[(\omega^{lk})_1\, n_l + \frac{1}{r}(\omega^{lk})_2\, m_l + \frac{1}{r\sin\theta}(\omega^{lk})_3\, l_l\right] \\ &+ \frac{1}{r}(\omega^{kl})_2\,(n_i m_l - m_i n_l) + \frac{1}{r\sin\theta}(\omega^{kl})_3\,(n_i l_l - l_i n_l) + O_3' \end{aligned}\right\} \qquad \text{(A.20)}$$

and

$$\left.\begin{aligned}\frac{1}{2} Z_k{}^k &= (\omega^{l\,4})_1\, n_l + \frac{1}{r}(\omega^{l\,4})_2\, m_l + \frac{1}{r\sin\theta}(\omega^{l\,4})_3\, l_l \\ &\quad + \frac{1}{2\,r}(\omega^{\varkappa\lambda})_2\,(n_\varkappa m_\lambda - m_\varkappa n_\lambda) \\ &\quad + \frac{1}{2\,r\sin\theta}(\omega^{\varkappa\lambda})_3\,(n_\varkappa l_\lambda - l_\varkappa n_\lambda) + O_3'.\end{aligned}\right\}\tag{A.21}$$

Hence,

$$\left.\begin{aligned}\varkappa Y_i{}^{4\,l}\mu_l &= Z_i{}^4 - \frac{1}{2} Z_k{}^k \delta_i^4 \\ &= \frac{1}{r}(\omega^{4\,\lambda})_2\,(n_i m_\lambda - m_i n_\lambda) + \frac{1}{r\sin\theta}(\omega^{4\,\lambda})_3\,(n_i l_\lambda - l_i n_\lambda) \\ &\quad - \delta_i^4\left[\frac{(\omega^{\varkappa\lambda})_2}{2\,r}(n_\varkappa m_\lambda - m_\varkappa n_\lambda) + \frac{(\omega^{\varkappa\lambda})_3}{2\,r\sin\theta}(n_\varkappa l_\lambda - l_\varkappa n_\lambda)\right] + O_3',\end{aligned}\right\}\tag{A.22}$$

which is the formula (6.24) in the text.

Next, we shall calculate the complex $\vartheta_i{}^k$ created in a completely empty space by the infinitesimal transformation (8.21). Since

$$\frac{\partial X^i}{\partial x^k} = \delta_k^i + \xi^i{}_{,\,k},\tag{A.23}$$

we get for the matrix tensor g_{ik} to the first order

$$\left.\begin{aligned}g^{ik} &= \frac{\partial X^l}{\partial x^i}\frac{\partial X^m}{\partial x^k}\eta^{lm} = \eta_{ik} + \xi_{i,\,k} + \xi_{k,\,i} \\ g^{ik} &= \eta^{ik} - \xi^{i,\,k} - \xi^{k,\,i}\end{aligned}\right\}\tag{A.24}$$

with

$$\xi_i = \eta_{ik}\,\xi^k, \quad \xi^{i,\,k} = \eta^{kl}\,\xi^i{}_{,\,l}.\tag{A.25}$$

To the same order we have

$$\left.\begin{aligned}\Gamma_{kl}^i &= \frac{\eta^{ir}}{2}(g_{rk,\,l} + g_{rl,\,k} - g_{kl,\,r}) = \xi^i{}_{,\,k,\,l} \\ \Gamma_{lm}^m &= \xi^r{}_{,\,r,\,l}.\end{aligned}\right\}\tag{A.26}$$

Further, since

$$\sqrt{-g} = 1 + \xi^r{}_{,\,r},\tag{A.27}$$

we get

$$\left(\sqrt{-g}\,g^{lm}\right)_{,\,i} = \eta^{lm}\,\xi^r{}_{,\,r,\,i} - \xi^{l,\,m}{}_{,\,i} - \xi^{m,\,l}{}_{,\,i}.\tag{A.28}$$

Thus, the Einstein Lagrangian (2.8) is to the second order

$$\mathfrak{L}_E = \xi^r{}_{,r,s} \Box \xi^s - \xi_{r,s,t} \xi^{t,s,r} \tag{A.29}$$

with

$$\Box \xi^i = \xi^i{}_{,k}{}^{,k}, \tag{A.30}$$

and for Einstein's energy-momentum complex (2.10) we get to the same order

$$\left.\begin{aligned} \vartheta_i{}^k = \frac{1}{2\varkappa}\{\xi^r{}_{,r,i} \Box \xi^k - 2\,\xi^k{}_{,l,m} \xi^{l,m}{}_{,i} - \xi^r{}_{,r}{}^{,k} \xi^s{}_{,s,i} \\ + (\xi^{k,m}{}_{,i} + \xi^{m,k}{}_{,i})\,\xi^r{}_{,r,m} - \delta_i^k \mathfrak{L}_E\} \end{aligned}\right\} \tag{A.31}$$

which in general does not vanish. Even if we require that the new system should be harmonic, we have in general $\vartheta_i{}^k \neq 0$. From (A.28) we see that the new system is harmonic if

$$(\sqrt{-g}\,g^{ik})_{,k} = -\Box \xi^i = 0, \tag{A.32}$$

which only will make the first terms in (A.29) and (A.31) disappear.

On the other hand, if $\xi^i = \xi^i(u)$ is a function of $u = x^4 - x^1$ only, so that the metric tensor (A.24) has the form of a plane wave, we have

$$\xi^i{}_{,k,l} = (\xi^i)''\,\bar{\mu}_k\,\bar{\mu}_l, \tag{A.33}$$

where $\bar{\mu}_k$ are the quantities defined by (8.9) for which

$$\bar{\mu}_k\,\bar{\mu}^k = 0. \tag{A.34}$$

In this case, we get at once from (A.29)–(A.34)

$$(\sqrt{-g}\,g^{ik})_{,k} = -\Box \xi^i = 0, \quad \mathfrak{L}_E = 0 \tag{A.35}$$

and

$$\vartheta_i{}^k = \frac{1}{2\varkappa}\{-(\xi^r)''\,\bar{\mu}_r\,\bar{\mu}^k\,(\xi^s)''\,\bar{\mu}_s\,\bar{\mu}_i + (\xi^m)''\,\bar{\mu}^k\,\bar{\mu}_i\,(\xi^r)''\,\bar{\mu}_r\,\bar{\mu}_m\} = 0. \tag{A.36}$$

Finally, we shall show that a rotation of the tetrads (8.13) with coefficients $\Omega_{(ab)}$ of the type (8.26) does not change the value (8.14) of the complex $\mathrm{t}_i{}^k$ defined by (2.31). To the first order, the rotated tetrads are

$$\check{h}_{(a)\,i} = \Omega_{(a)}{}^{(b)}\,h_{(b)\,i} = \eta_{ai} + \frac{1}{2}\overset{(1)}{y}_{ai} + \bar{\omega}_{ai}. \tag{A.37}$$

Since both $\overset{(1)}{y}_{ai}$ and $\overline{\omega}_{ai}$ are functions of $\bar{u} = \bar{l} - \bar{x}$ only, we get to the same order

$$\check{\gamma}_{ikl} = \check{h}_i^{(a)}\, \check{h}_{(a)\, k:\, l} = \left[\frac{1}{2}\, \overset{(1)}{y}_{kl}\, \bar{\mu}_i - \frac{1}{2}\, \overset{(1)}{y}_{il}\, \bar{\mu}_k + \overline{\omega}_{ik}\, \bar{\mu}_l\right]'. \tag{A.38}$$

Here, $\bar{\mu}_i$ are the constant numbers defined by (8.9), and the prime means differentiation with respect to $\bar{u}$. Similarly, we get

$$\left.\begin{aligned} \check{\Delta}^i{}_{kl} &= \check{h}^{(a)\, i}\, \check{h}_{(a)\, k,\, l} = \left[\frac{1}{2}\, \overset{(1)}{y}{}^i_k + \overline{\omega}^i{}_k\right]' \bar{\mu}_l \\ \check{\Phi}_k &= \check{\gamma}^i{}_{ki} = [\overline{\omega}^i{}_k\, \bar{\mu}_i]', \end{aligned}\right\} \tag{A.39}$$

where we have used the relations

$$\overset{(1)}{y}_{ik}\, \bar{\mu}^k = 0\,, \qquad \overset{(1)}{y}{}^i_i = 0 \tag{A.40}$$

following from (8.12).

By means of these expressions we can now calculate the Lagrangian (2.29) corresponding to the rotated tetrads to the second order, which gives

$$\left.\begin{aligned} \check{\mathfrak{L}} &= \check{\gamma}_{ikl}\, \check{\gamma}^{lki} - \check{\Phi}_k\, \check{\Phi}^k \\ &= \left[\frac{1}{2}\, \overset{(1)}{y}_{kl}\, \bar{\mu}_i - \frac{1}{2}\, \overset{(1)}{y}_{il}\, \bar{\mu}_k + \overline{\omega}_{ik}\, \bar{\mu}_l\right]' \left[\frac{1}{2}\, \overset{(1)}{y}{}^{ki}\, \bar{\mu}^l - \frac{1}{2}\, \overset{(1)}{y}{}^{li}\, \bar{\mu}^k + \overline{\omega}^{lk}\, \bar{\mu}^i\right]' \\ &\quad - [\overline{\omega}^i{}_k\, \bar{\mu}_i]'\, [\overline{\omega}^{lk}\, \bar{\mu}_l]' \\ &= (\overline{\omega}_{ik})'\, (\overline{\omega}^{lk})\; \bar{\mu}^i\, \bar{\mu}_l - (\overline{\omega}^i{}_k)'\, (\overline{\omega}^{lk})'\, \bar{\mu}_i\, \bar{\mu}_l = 0\,. \end{aligned}\right\} \tag{A.41}$$

Similarly, by means of (2.31),

$$\left.\begin{aligned} \varkappa \check{\mathfrak{t}}_i{}^k &= \check{\gamma}^{km}{}_l\, \Delta^l{}_{mi} - \check{\Phi}^l\, \check{\Delta}^k{}_{li} + \check{\Delta}^l{}_{li}\, \check{\Phi}^k \\ &= \left[\frac{1}{2}\, \overset{(1)}{y}{}^m_l\, \bar{\mu}^k - \frac{1}{2}\, \overset{(1)}{y}{}^k_l\, \bar{\mu}^m + \overline{\omega}^{km}\, \bar{\mu}_l\right]' \left[\frac{1}{2}\, \overset{(1)}{y}{}^l_m + \overline{\omega}^l{}_m\right]' \bar{\mu}_i \\ &\quad - [\overline{\omega}^{rl}\, \bar{\mu}_r]' \left[\frac{1}{2}\, \overset{(1)}{y}{}^k_l + \overline{\omega}^k{}_l\right]' \bar{\mu}_i + \left[\frac{1}{2}\, \overset{(1)}{y}{}^l_l + \overline{\omega}^l{}_l\right]' \bar{\mu}_i\, [\overline{\omega}^{rk}\, \bar{\mu}_r]' \\ &= \left[\frac{1}{4}\left(\overset{(1)}{y}_{lm}\right)' \left(\overset{(1)}{y}{}^{lm}\right)' + \frac{1}{2}\left(\overset{(1)}{y}{}^{lm}\right)' (\overline{\omega}_{lm})'\right] \bar{\mu}_i\, \bar{\mu}^k \\ &\quad + (\overline{\omega}^l{}_l)'\, (\overline{\omega}^{rk})'\, \bar{\mu}_i\, \bar{\mu}_r. \end{aligned}\right\} \tag{A.42}$$

Here we have again used the relations (A.40). Finally, we shall make use of the fact that $\overline{\omega}_{lm}$ is antisymmetrical in l and m, i.e.

$$\bar{\omega}_{lm} = -\bar{\omega}_{ml}, \quad \omega'^{l}{}_{l} = 0, \tag{A.43}$$

while $\overset{(1)}{y}{}^{lm}$ is symmetrical. Therefore, since

$$\Big(\overset{(1)}{y}_{lm}\Big)'\Big(\overset{(1)}{y}{}^{lm}\Big)' = 8\,\bar{c}'(\bar{u})^2, \tag{A.44}$$

we get for the complex $\check{\mathfrak{t}}_i{}^k$

$$\check{\mathfrak{t}}_i{}^k = \frac{2}{\varkappa}\bar{c}'(\bar{u})^2\,\bar{\mu}_i\,\bar{\mu}^k = \mathfrak{t}_i{}^k, \tag{A.45}$$

which completes the proof of the invariance of the energy-momentum complex under the rotations of the type (8.26).

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